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VOLUME XII

SECTION A

PUBLISHED BY THE COUNCIL

CONTENTS

SECTION A

MATHEMATICS

PHYSICS

CHEMISTRY

INDEX—XII(A)

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PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
INDIA

1942

VOLUME XII

SECTION B

PUBLISHED BY THE COUNCIL

CONTENTS

SECTION B

ZOOLOGY

BOTANY

INDEX—XII(B)

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ON POLYTROPES ROTATING WITH VARIABLE ANGULAR VELOCITY (IN TIME)

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Communicated by Prof. A. C. Banerji
(Received on November 21, 1941)

§1. Considerations of gravitational equilibrium of fluid masses under given conditions of rotation and oscillation have become important in comparatively recent times. Emden's work on non-rotating polytropic configurations helped Eddington's pioneer investigations of the case of pulsating gaseous spheres with application to the important considerations of Cepheid Variables, and the work of Chandrasekhar in rotating polytropes. Bhatnagar's work on polytropes rotating with a variable angular velocity depending on the distance from the axis of rotation suggested the introduction of the time factor into the variation of the angular velocity. Besides the applicability of such an investigation to the pulsation theory of the Cepheid Variables, the consideration of the oscillations of a rotating polytrope is made more important by a recent suggestion of Prof. Banerji about the possibility of its application to the Theory of the Origin of Planets.

The consideration of the combined oscillation and rotation of fluid masses of non-constant density is necessarily very complex being more difficult than the obviously simplified case of the constancy of the density when the gravitational potential of the mass is easily put down, at the surface of the fluid mass, in terms of the normal displacement at the surface, while the equation of continuity is very simply satisfied. Here we have considered the consequences in a few cases of variable angular velocity, varying with the time. Consideration of the Pulsation Theory of the Cepheids, it is hoped, will be given in another communication.

In this communication we shall consider the dynamical equations of motion, together with the Equation of Continuity and Poisson's Equation connecting the

density with the gravitational potential of the mass under a given polytropic relationship. We shall deduce the conditions of integrability and proceed to the integration of the equations of motion after suitable choice to satisfy these conditions.

§2. The general equations of motion in polar co-ordinates of a fluid mass, the viscous forces being neglected, are

$$\frac{Dq_r}{Dt}-\frac{q_\theta^2+q_\phi^2}{r}=\frac{\partial V}{\partial r}-\frac{1}{\rho}\frac{\partial p}{\partial r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{Dq_\theta}{Dt}-\frac{q_\phi^2\cot\theta-q_r\,q_\theta}{r}=\frac{\partial V}{r\partial\theta}-\frac{1}{\rho}\frac{\partial p}{r\partial\theta} \quad . \quad . \quad . \quad (2)$$

$$\frac{Dq_\phi}{Dt}+\frac{q_r\,q_\phi+q_\theta\,q_\phi\cot\theta}{r}=\frac{\partial V}{r\sin\theta\partial\phi}-\frac{1}{\rho}\frac{\partial p}{r\sin\theta\partial\phi} \quad . \quad . \quad . \quad (3)$$

where

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+q_\theta\frac{\partial}{r\partial\theta}+q_\phi\frac{\partial}{r\sin\theta\partial\phi} \quad . \quad . \quad . \quad . \quad (4)$$

the equation of continuity being

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_r r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho q_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial(\rho q_\phi)}{\partial\phi}=0 \quad . \quad (5)$$

Here we consider the case when the fluid mass is rotating about the axis $\theta=0$ with an angular velocity ω which is a function of the time, the radius and the latitude, i.e.,

$$\omega=\omega\,(r,\mu,t)$$

μ being written for $\cos\theta$, and we shall superpose on this motion a radial velocity q_r which we take like ω as a function of r, μ and t, i.e.,

$$q_r=q_r\,(r,\mu,t),\quad \rho=\rho\,(r,\mu,t)$$

and

$$q_\phi=r\sin\theta.\omega=r\sin\theta.\omega\,(r,\mu,t)$$

Under the conditions as taken above, the gravitational potential will be independent of the longitude so that we can write Poisson's Equation

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left(\overline{1-\mu^2}\frac{\partial V}{\partial\mu}\right)\equiv\Delta^2V=-4\pi r\rho$ and the Equation of Continuity in the form

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho q_r r^2\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)'$$

We assume the polytropic relationship between the pressure p, and the density ρ

$$p=K\rho^{1+\frac{1}{n}}$$

and replace the function ρ by the function Θ where

$$\rho=\alpha\Theta^n$$

The equations of motion then take the forms

$$\frac{\partial q_r}{\partial t}+q_r\frac{\partial q_r}{\partial r}-\omega^2 r\ \sin^2\theta=\frac{\partial \mathrm{V}}{\partial r}-\mathrm{K}\alpha^{\frac{1}{n}}\ (n+1)\frac{\partial \Theta}{\partial r}\ . \qquad (1)'$$

$$-r\omega^2\ \sin\theta\ \cos\theta=\frac{\partial \mathrm{V}}{r\partial\theta}-\mathrm{K}\alpha^{\frac{1}{n}}(n+1)\frac{\partial \Theta}{r\partial\theta}. \qquad (2)'$$

$$r\ \sin\theta\frac{\partial\omega}{\partial t}+q_r\frac{\partial\omega}{\partial r}r\sin\theta+2q_r\ \omega\sin\theta=0 \qquad (3)'$$

It directly follows from equation (3)$'$ that

$$q_r=-\frac{r}{2}\ \frac{1}{\omega}\ \frac{\partial\omega}{\partial t}.\ \frac{1}{1+\frac{r}{2}\frac{1}{\omega}\frac{\partial\omega}{\partial r}} \qquad (7)$$

Changing from θ to μ we get from (1)$'$ and (2)$'$

$$\frac{\partial q_r}{\partial t}+q_r\frac{\partial q_r}{\partial r}-\omega^2 r+\omega^2 r\mu^2=\frac{\partial}{\partial r}(\mathrm{V}-\mathrm{R}\Theta)\ . \qquad (1)''$$

where
$$\mathrm{R}=(n+1)\mathrm{K}\alpha^{\frac{1}{n}}$$

$$r^2\omega^2\mu \qquad =\frac{\partial}{\partial\mu}(\mathrm{V}-\mathrm{R}\Theta)\ . \qquad (2)''$$

Our solutions will be those of (1)$''$ and (2)$''$ with the help of (7) such as to satisfy (5)$'$.

§3. *Condition of Integrability :* In order that the two equations (1)$''$ and (2)$''$ may be integrable simultaneously we have to satisfy the condition

$$\frac{\partial}{\partial\mu}\left\{\frac{\partial q_r}{\partial t}+q_r\frac{\partial q_r}{\partial r}-\omega^2 r+\omega^2 r\,\mu^2\right\}=\frac{\partial}{\partial r}(r^2\omega^2\mu)$$

Remembering that q_r and ω are functions of μ and r, the above relation reduces to

$$\frac{\partial^2 q_r}{\partial t\partial\mu}+\frac{\partial}{\partial\mu}\left(q_r\frac{\partial q_r}{\partial r}\right)-\frac{\partial}{\partial\mu}(+\omega^2 r)+2\omega^2 r\mu+\mu^2 r\frac{\partial}{\partial\mu}\omega^2$$

$$=\quad 2r\mu\omega^2+r^2\mu\frac{\partial}{\partial r}(\omega^2)$$

that is,

$$\frac{\partial^2 q_r}{\partial t\partial\mu}+\frac{\partial}{\partial\mu}\left(q_r\frac{\partial q_r}{\partial r}\right)-r(1-\mu^2)\frac{\partial}{\partial\mu}(\omega^2)=r^2\mu\frac{\partial}{\partial r}(\omega^2) \qquad (8)$$

§4. *Approximations:* The above condition (8) by substitution from equation (7) for q_r will give us a partial differential equation in ω of the third order being of the first order in r and μ, and of the second order in the time t. This is rather complicated to handle. We therefore introduce function ω_1 which also depends on time such that

$$\omega=\omega_0\ (1+\omega_1) \qquad \qquad (9)$$

where ω_0 is the static value of ω. We substitute (9) in (8) and neglect squares and higher powers of ω_1. First we write down

$$\omega\ =\omega_0+\omega_0\ \omega_1$$

$$q_r=-\frac{r}{2}\frac{\partial\omega_1}{\partial t}$$

$$\omega^2=\omega_0{}^2+2\omega_0{}^2\omega_1$$

Then eqn. (8) takes the form

$$-\frac{r}{2}\frac{\partial^2}{\partial t^2}\frac{\partial\omega_1}{\partial\mu}-r\,(1-\mu^2).\ 2\omega_0{}^2\frac{\partial\omega_1}{\partial\mu}=r^2\mu.\ 2\omega_0{}^2\frac{\partial\omega_1}{\partial r}$$

Dividing the above by $2\omega_0{}^2r^2\mu$ we write the equation in the form

$$-\frac{1}{4\omega_0{}^2}\frac{\partial^2}{\partial t^2}\frac{\partial\omega_1}{r\mu\partial\mu}-\frac{1-\mu^2}{r\mu}\frac{\partial\omega_1}{\partial\mu}=\frac{\partial\omega_1}{\partial r}\ .$$

§5. It is easily seen that in (8)$'$ if we put $\frac{\partial\omega_1}{\partial t}=\frac{\partial\omega_1}{\partial r}=0$, $\frac{\partial\omega_1}{\partial\mu}$ is necessarily zero in order that the condition may be satisfied, that is, it shows that ω_1 cannot be a function of μ alone. Similarly, in the case $\frac{\partial\omega_1}{\partial t}=\frac{\partial\omega_1}{\partial\mu}=0$, $\frac{\partial\omega_1}{\partial r}$ is found to be zero showing that the amplitude also cannot be a function of r alone.

Again, if we assume that $\frac{\partial\omega_1}{\partial\mu}=0$, a necessary condition for integrability is that $\frac{\partial\omega_1}{\partial r}=0$; we take this case, that is, when ω_1 is purely a function of the time. It will then be observed that the equations (1), (2) and (3) reduce to

$$\frac{\partial q_r}{\partial t}+q_r\frac{\partial q_r}{\partial r}-\omega^2 r+\omega^2 r\mu^2=\frac{\partial}{\partial r}\,(\mathrm{V}-\mathrm{R}\,\Theta)\ . \qquad (\mathrm{A})$$

$$r^2\omega^2\mu=\frac{\partial}{\partial\mu}\,(\mathrm{V}-\mathrm{R}\,\Theta)\ . \qquad (\mathrm{B})$$

$$q_r=-\frac{r}{2}\frac{1}{\omega}\frac{d\omega}{dt}\ . \qquad (\mathrm{C})$$

Since $\frac{1}{\omega}\frac{d\omega}{dt}$ is purely a function of the time t, a direct consequence of our assumption is that the radial velocity is proportional to the distance from the origin and

that during the time in which ω is increasing the radial velocity is inwards, and the reverse is the case when ω is decreasing.

In consequence of equation (6), the equations (A) and (B) become

$$\left\{-\frac{r}{2}\frac{d}{dt}\left(\frac{1}{\omega}\frac{d\omega}{dt}\right)+\frac{r}{4}\frac{1}{\omega^2}\left(\frac{d\omega}{dt}\right)^2\right\}-r\omega^2(1-\mu^2)=\frac{\partial V}{\partial r}-K\alpha^{\frac{1}{n}}(n+1)\frac{\partial\Theta}{\partial r}$$

and

$$r\omega^2\mu=\frac{\partial V}{r\partial\mu}-\frac{K\alpha^{\frac{1}{n}}(n+1)}{r}\frac{\partial\Theta}{\partial\mu}$$

from which we may write

$$\frac{\partial V}{\partial r}=K\alpha^{\frac{1}{n}}(n+1)\frac{\partial\Theta}{\partial r}-r\left\{\omega^2(1-\mu^2)+\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\omega}\frac{d\omega}{dt}\right)-\frac{1}{4}\frac{1}{\omega^2}\left(\frac{d\omega}{dt}\right)^2\right\}$$

$$\frac{\partial V}{\partial\mu}=K\alpha^{\frac{1}{n}}(n+1)\frac{\partial\Theta}{\partial\mu}+r^2\omega^2\mu$$

Substituting these values of $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial\mu}$ in Poisson's density-potential relation we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2K\alpha^{\frac{1}{n}}(n+1)\frac{\partial\Theta}{\partial r}-r^3\left\{\omega^2(1-\mu^2)+\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\omega}\frac{d\omega}{dt}\right)-\frac{1}{4}\left(\frac{1}{\omega}\frac{d\omega}{dt}\right)^2\right\}\right]$$

$$+\frac{1}{r^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)K\alpha^{\frac{1}{n}}(n+1)\frac{\partial\Theta}{\partial\mu}+(1-\mu^2)r^2\omega^2\mu\right]=-4\pi\gamma\alpha\Theta^n$$

Putting $4\pi\gamma r^2=(n+1)K\alpha^{\frac{1}{n}-1}\xi^2$, and $\omega^2=2\pi\gamma\alpha.v$, the above equation simplifies to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)=-\Theta^n+v+\frac{3}{4\pi\gamma\alpha}\left\{\frac{1}{4}\frac{d}{dt}\left(\frac{1}{v}\frac{dv}{dt}\right)-\frac{1}{16}\left(\frac{1}{v}\frac{dv}{dt}\right)^2\right\}$$

We can however note that the terms in these brackets are functions of the time t, and we can write $\Delta v\times 4\pi\gamma\alpha/3$ for this term so that Δv is a function of the time only.

Denoting by R the expression $K\alpha^{\frac{1}{n}}(n+1)$, we have by integration of (A) and (B) and changing from ω to v and from r to ξ, the equations

$$V-R\Theta=-R\frac{\xi^2}{4}\left[(1-\mu^2)v+\left\{\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\omega}\frac{d\omega}{dt}\right)^2-\frac{1}{16}\left(\frac{1}{v}\frac{dv}{dt}\right)^2\right\}\frac{1}{2\pi\gamma\alpha}\right]+f_1(\mu,t)$$

and

$$V-R\Theta=\frac{R\xi^2}{4}v\mu^2+f_2(\xi,t)$$

Writing $-\frac{1}{12}v\mathrm{R}\xi^2+f_3(\xi,t)$ in place of the arbitrary $f_2(\xi,t)$ we get

$$\mathrm{V}-\mathrm{R}\Theta\equiv\frac{\mathrm{R}v\xi^2}{6}\{\mathrm{P}_2(\mu)-1\}-\frac{\mathrm{R}\xi^2}{4}\,\frac{4\pi\gamma\alpha}{3}.\ \triangle v\frac{1}{2\pi\gamma\alpha}+f_1(\mu,t)$$

$$\equiv\frac{\mathrm{R}v\xi^2}{6}\{\mathrm{P}_2(\mu)-1\}+f_3(\xi,t)$$

Since the identity holds for all values of ξ, μ and t, we have

$$f_1(\mu,t)=-\frac{\mathrm{R}\xi^2}{6}\triangle v+f_3(\xi,t)$$

The right-hand side here is a function of ξ and t only, whereas the left-hand side is a function of μ and t only. Therefore both are merely equal to a function $f(t)$ of t. Hence, we get

$$\mathrm{V}-\mathrm{R}\Theta=\frac{\mathrm{R}v\xi^2}{6}\{\mathrm{P}_2(\mu)-1\}-\frac{\mathrm{R}\xi^2}{6}\triangle v+f(t)$$

Using the definition of $\triangle v$ as above, we can write the above in the form

$$\mathrm{V}-\mathrm{R}\Theta=-\mathrm{R}\frac{\xi^2}{6}(v+\triangle v)+\mathrm{R}\frac{\xi^2}{6}v\mathrm{P}_2(\mu)+f(t) \quad . \quad . \quad . \quad (11)$$

The equation of continuity simply becomes

$$\frac{\partial\rho}{\partial t}-\frac{1}{2r^2}\frac{\partial}{\partial r}\left(\rho r^3\frac{1}{\omega}\frac{d\omega}{dt}\right)=0$$

Putting $\alpha\Theta^n$ for ρ and changing from r to ξ, we write the equation of continuity in the form

$$\frac{\partial\Theta}{\partial t}-\frac{1}{v}\frac{dv}{dt}\frac{1}{4}\left\{\frac{3\Theta}{n}+\xi\frac{\partial\Theta}{\partial\xi}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The Poisson equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)=-\Theta^n+v+\triangle v \quad . \quad . \quad (13)$$

It is directly seen that equations (11) and (13), $(v+\triangle v)$ not being a function of μ or ξ, are of the same form as Chandrasekhar's equations for statical solutions of slowly rotating polytropes, their solutions being therefore expressible in terms of Emden's function $\theta(\xi)$ and auxiliary Emden's functions $\psi_2(\xi)$ and $\psi_0(\xi)$ in the form

$$\Theta=\theta(\xi)+v\{\psi_0(\xi)+\mathrm{A}_2\psi_2(\xi)\mathrm{P}_2(\mu)\}+\triangle v.\psi_0(\xi) \quad . \quad . \quad . \quad (14)$$

The term in Θ in equation (14), dependent on the time, is

$$\psi_0(\xi).\ \triangle v$$

It is therefore easily seen that the equation of continuity (12) remains unsatisfied under our assumptions, unless $\frac{d\omega}{dt}=0$. We therefore have to conclude that purely radial oscillations of a rotating polytrope are not possible.

§6. We now take the case $\frac{\partial \omega_1}{\partial r}=0$, that is, ω_1 does not depend on the distance from the origin. The variation is now depending on conical surfaces with the axis of rotation as their axis and the centre the vertex. The condition of integrability becomes in this case

$$-\frac{1}{4\omega_0^2}\frac{\partial^2}{\partial t^2}\frac{\partial \omega_1}{r\mu\partial\mu}-\frac{1-\mu^2}{r\mu}\frac{\partial\omega_1}{\partial\mu}=0$$

i.e.,
$$\left(\frac{\partial^2}{\partial t^2}+4\omega_0^2\nu\right)\frac{\partial\omega_1}{\partial\nu}=0 \qquad \nu=1-\mu^2$$

Such a condition of integrability is satisfied by a function of the form

$$\omega_1(\nu, t)=\int A(\nu)\cos(2\omega_0\sqrt{\nu}\,t)\,d\nu+\int B(\nu)\sin(2\omega_0\sqrt{\nu}\,t)\,d\nu+C(t)$$

For example, in the simple case $B(\nu)\equiv C(t)\equiv 0$ and $A(\nu)=\frac{a}{2}\nu^{-\frac{1}{2}}$ we get

$$\omega_1(\nu, t)=a\frac{\sin 2\omega_0\sqrt{\nu}\,t}{2\omega_0 t}$$

It shows that the amplitude goes on decreasing towards zero as the time increases. It indicates that if such an oscillation be started it will go on decreasing until eventually it assumes Chandrasekhar's static form. It is only an indication, however, since we have not completely solved all the equations of this case.

§7. Lastly we take the case $\frac{\partial\omega_1}{\partial t}=0$, that is, the variation does not depend upon the time. Then

$$-(1-\mu^2)\frac{\partial\omega_1}{\mu\partial\mu}=\frac{\partial\omega_1}{\partial r}$$

which is satisfied by taking $\omega_1=\omega_1\,(r\sqrt{1-\mu^2})$. It is easily verified, however, that in this case $\omega_1=\omega_1\,(r\sqrt{1-\mu^2})$ satisfies the condition of integrability without any appeal to approximation. Further in this case q_r is obviously zero Hence, the equation of continuity is identically satisfied. Thus having satisfied, firstly the equation of continuity, secondly the third equation of motion, and thirdly the condition of integrability, we can proceed to the integration of the first two equations of motion and with the help of Poisson's equation connecting the gravitational potential with the density deduce Θ. In particular we may consider $\omega=\omega_0(1+\lambda r\sqrt{1-\mu^2})$. Chandrasekhar considered $\omega_1=0$. Bhatnagar considered $\omega=\frac{\omega_0}{r\sqrt{1-\mu^2}}$ with certain modifications, and the case $\omega'=\omega_0^2\{1-\alpha\frac{r^2}{R_s^2}(1-\mu^2)\}$.

§8. We now come to a different consideration of the condition of integrability (8)′

$$-\frac{1}{4\omega_0^2}\frac{\partial^2}{\partial t^2}\frac{\partial\omega_1}{r\mu\partial\mu}-\frac{1-\mu^2}{r\mu}\frac{\partial\omega_1}{\partial\mu}=\frac{\partial\omega_1}{\partial r}$$

We find that if we assume ω_1 to be harmonic with period $\frac{2\pi}{\sigma}$ the condition becomes

$$\left[\frac{\sigma^2}{4\omega_0{}^2}-(1-\mu^2)\right]\frac{\partial\omega_1}{\partial\mu.\mu}=\frac{\partial\omega_1}{\partial r}$$

Hence, our condition of integrability is satisfied if ω_1 is a function of $r^2(\lambda^2-\sin^2\theta)$ where $\lambda^2=\frac{\sigma^2}{4\omega_0{}^2}$, i.e., we write the angular velocity in the form

$$\omega=\omega_0\{1+\omega_1(r^2[\lambda^2-\sin^2\theta])\cos\sigma t\} \quad . \quad . \quad . \quad . \quad (15)$$

In the case when λ is zero this as we know reduces to $\omega=\omega_0(1+\omega_1(r\sin\theta))$. The amplitude function can be expanded in negative powers of r, the first term of which may be written $\frac{a_1}{r\sqrt{\lambda^2-\sin^2\theta}}$. On the surface with r constant, this is a maximum for $\theta=\frac{\pi}{2}$ and decreases as $\theta\to 0$, i.e., it is maximum on the equatorial plane and decreases towards the poles, which is similar to the variation of the surface rotation of the sun which has not yet been satisfactorily taken account of in any theory of steady angular velocity. It will be observed, however, that if in the above equation (15) we put $\cos\sigma t=0$, we get $\omega=\omega_0$, i.e., periodically the sun will move with uniform rotation. In the absence of a complete data to bear out the fact, we cannot say anything further. Concerning Bhatnagar's work on variable angular velocity Prof. A. C. Banerji remarks** that "In the variation law taken for ω containing only one term, we find that the angular velocity increases with latitude. But in the case of the sun it is observed that the rotation of its surface is maximum in the equatorial plane and decreases towards the poles This is apparently contradictory to our assumptions, but it may be remarked that the equatorial acceleration can be accounted for by taking two or more terms of the series for the value of ω and by suitably adjusting the constants a_1, a_2, a_3. . . . . some of which may be negative"

But here ω has been taken as a function of $r\sin\theta$ and we can write

$$\omega=f(r\sin\theta)$$

If $f(r\sin\theta)$ increases as $r\sin\theta$ increases then ω would have its maximum value at the equator. In this case the inner parts of the sun would have a lesser speed of rotation than the outer. So on any theory of steady but non-uniform angular velocity simultaneous increase in the angular velocity as the latitude decreases and as we proceed further from the axis of rotation cannot be explained.

I am grateful to Prof. A. C. Banerji for his keen interest in the work.

**Presidential Address, Section of Mathematics, Proceedings of the Twenty-seventh Indian Science Congress 1940, p. 35.

NOTE ON A THEOREM OF VALIRON AND COLLINGWOOD

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(Received on December 2, 1941)

G. Valiron and E. F. Collingwood proved that[1]

THEOREM A. *If* f(z) *be any Integral function of positive order* $\rho<1$, *then*

$$\liminf_{r\to\infty} \frac{\mu N(r, a)}{\log M(r^{\mu})} \geq 1$$

for all values of a *and all values of* $\mu<1-\rho$.

They also proved an analogous result for functions of zero order and deduce a known result.

THEOREM B. *If for* f(z)

$$\log M(r)=O((\log r)^2)$$

then

$$\log M(r)\sim N(r, a)$$

for every a.

In this note I show that the results of Theorems A and B are in a sense the best possible. I prove

THEOREM 1. *Given any non-decreasing function* $\theta(x)$ *tending to infinity with* x, *there exists an Integral function* f(z) *for which*

$$\lim_{r\to\infty} \frac{\log M(r)}{(\log r)^2\theta(r)}=0 \tag{1}$$

$$\limsup_{r\to\infty} \frac{\log M(r)}{N(r, 0)}=\infty \tag{2}$$

THEOREM 2. *Given any* $\rho\geq 0$, *and* $\psi(x)$ *any positive function for which*

$$\limsup_{x\to\infty} \frac{\log \psi(x)}{\log x}\leq\rho$$

(1) G. Valiron and E. F. Collingwood, London Math. Soc. J. 4. (1929), 210—213.

M(r), N(r,a) $n(r,a)$, and T(r) have their usual meanings, See E. C. Titchmarsh Theory of Functions (Second Edition), Chapter 8.

there exists an Integral function F(z) *for which*

(3) $$\liminf_{r\to\infty} \frac{R\,N(r, o)}{\log M(r^{1-\rho/k})} = 0$$

(4) $$\limsup_{r\to\infty} \frac{\log M(r/R)}{N(r, o)\psi(r)} = \infty$$

(5) $$\limsup_{r\to\infty} \frac{\log M(r/R)}{n(r, o)\psi(r)} = \infty$$

where $k = 1 + [\rho]$, $R = \exp\ (\log r/\theta(r))$ *and* $\theta(r)$ *has been defined in the statement of Theorem* 1.

2. Lemma 1. *Given* $\theta(x)$, *a non-decreasing function tending to* ∞ *with* x, *there exists a non-decreasing function* $\Psi(x)$ *tending to infinity with* x *and such that*

$$\Psi(x) \leq l_3 x = \log\log\log x\,;\ \Psi(x) \leq \sqrt{\theta(x)} \text{ for all } x \geq X$$

Proof. Choose $X \geq e^e$ such that $\theta(x) > 0$. Let

$$\Psi(x) = 0 \text{ for } 0 \leq x \leq X$$
$$= l_3\, x - l_3 X \text{ for } X < x \leqq X_1$$

where X_1 is the first meet of the curve $y = \sqrt{\theta(x)}$ with $y = l_3\ x - l_3 X$ in the interval $x > X$. If they do not meet at all then we take

$\Psi(x) = l_3 x - l_3 X$ for $x > X$. If they meet, let

$\Psi(x) = l_3 X_1 - l_3 X$ for $X_1 \leq x \leqq 1 + \zeta_1$

where ζ_1 $(\geq X_1)$ is the greatest number such that

$\theta(\zeta_1) = \theta(X_1)$.

$\Psi(x) = l_3 x - l_3\ (1+\zeta_1) + l_3 X_1 - l_3 X$ for $1 + \zeta_1 < x \leq X_2$.

where X_2 is the first meet of $y = \sqrt{\theta\,(x)}$ with $y = l_3 x - l_3\ (1+\zeta_1) + l_3 X_1 - l_3 X$ in the interval $x > 1 + \zeta_1$. We repeat.

It is seen that for $x \geq X$

$\Psi\,(x) \leq l_3 x\,;\ \Psi\,(x) \leq \sqrt{\theta(x)}$ and $\Psi\ (x)$ is positive non-decreasing function tending to ∞ with x.

3. Proof of Theorem 1. Let $\lambda_1 = X$, λ_2 be defined by

$$\Psi\ (e^{\lambda_2}) = \text{Max}\ \{\lambda_1{}^2, \lambda_1{}^2\Psi(e^{\lambda_1})\}$$

and for $n \geq 3$, λ_n be defined by

$$\Psi\ (e^{\lambda_n}) = \lambda^2{}_{n-1}\Psi\ (e^{\lambda_{n-1}})$$

where X and $\Psi\ (x)$ have been defined in lemma 1. Let

$$f(z) = \prod_1^\infty \left\{ 1 + \frac{z}{e^{\lambda_n}} \right\}^{\lambda_n t_n}$$

where $t_n = [\Psi\ (e^{\lambda_n})]$. Since

$$\sum \frac{\lambda_n t_n}{e^{a\lambda_n}} \leq \sum \frac{\lambda_n\ \Psi(e^{\lambda_n})}{e^{a\lambda_n}} \leq \sum \frac{\lambda_n\ l_2\ \lambda_n}{e^{a\lambda_n}}$$

is convergent for every a >0, hence f (z) is an integral function of zero order. Let

$e^{\lambda_n} \leq r < e^{\lambda_{n+1}}$. Then $\log M\ (r) = \sum_1^{\infty} \lambda_n t_n \log \left(1+ \frac{r}{e^{\lambda_n}}\right)$

$$< n\lambda_{n-1}\ t_{n-1} \log\ (2r) + \lambda_n\ t_n \log\ (2r) + \frac{\lambda_{n+1} t_{n+1} r}{e^{\lambda_{n+1}}} + 0(1$$

$$= \quad A_1 \qquad\qquad + \quad A_2 \quad + \quad A_3 \quad + 0(1)$$

where for $n \geq n_0$

$$A_1 < 2n\lambda_{n-1} t_{n-1} \log r < 2n\lambda_{n-1} l_2 \lambda_{n-1} \log r$$

$$< \lambda^2_{n-1} \log r < l_2 \lambda_n \log r = 0\ (\ (\log r)^2)$$

$$A_2 < 2\lambda_n\ t_n \log r \leq 2\log^2 r\ \sqrt{\theta(e^{\lambda_n})} \leq 2\log^2 r\ \sqrt{\theta(r)}$$

$$A_3 < \frac{\lambda_{n+1}\ \Psi(e^{\lambda_{n+1}}) r}{e^{\lambda_{n+1}}}$$

If now $\quad e^{\lambda_n} \leq r \leq e^{\lambda_{n+1}}/2$ then

$$A_3 < \frac{\lambda_{n+1} l_2\ \lambda_{n+1} \cdot e^{\lambda_{n+1}}/2}{e^{\lambda_{n+1}}} = 0(1)$$

If $\quad e^{\lambda_{n+1}}/2 < r < e^{\lambda_{n+1}}$ then

$$A_3 < \lambda_{n+1}\ l_2\ \lambda_{n+1} = 0\ ((\log r)^2)$$

Hence,

$$\lim_{r \to \infty} \frac{\log M(r)}{(\log r)^2\ \theta(r)} = 0$$

Zeros of f(2) are $-e^{\lambda_n}$; let $r = e^{\lambda_n} - 1$. Then

$$N\ (r, o) = \int_{\lambda_1}^{r} \frac{n(x)dx}{x} < n(r)\ (\log r - \log \lambda_1) \sim \lambda_{n-1}\ t_{n-1}\ \lambda_n; \quad \log M(r/2)$$

$$> \lambda_n\ t_n \log \left\{ 1 + \frac{e^{\lambda_n} - 1}{2e^{\lambda_n}} \right\} > \frac{\lambda_n\ t_n}{4} \qquad \text{for } n \geq n_0.$$

Hence, for $n \geq n_1 > n_0$

$$\frac{\log M(r/2)}{N\ (r, o)} > \frac{\lambda_n\ t_n}{8\ \lambda_{n-1} t_{n-1} \lambda_n} \sim \frac{\Psi(e^{\lambda_n})}{8\ \lambda_{n-1}\ t_{n-1}} \sim \frac{\lambda_{n-1}}{8}$$

Hence,

$$\limsup_{r\to\infty} \frac{\log M(r)}{N(r, o)} = \limsup_{r\to\infty} \frac{T(r)}{N(r, o)} = \infty$$

Proof of Theorem 2. Let

$\psi(x) = \exp\{(\rho+\eta^1_x)\log x\}$ where we may suppose $\eta^1_x > 0$.

Let

$$\eta_r = \text{Max}\left\{\begin{matrix}\text{upper bound} \\ t \geq r\end{matrix}\ \eta^1_t,\ \frac{1}{\theta(r)}\right\}$$

Then η_r tends monotonically to zero as $r\to\infty$, and

$\psi(r) \leq \exp\{(\rho+\eta_r)\log r\}$. Let $\lambda_n = n^{n^n}$ $(n \geq 1)$; $\varepsilon_1 = \varepsilon_2 = 1$;

and

$$\varepsilon_n = 3\eta_n + \frac{(4+\rho)\log\lambda_{n-1}}{\log\lambda_n} \qquad n = 3, 4, 5, \ldots.$$

$$\zeta_n = [\lambda_n^{\rho/k+\varepsilon_n}] \qquad n = 1, 2, \ldots\ldots;\ F(z) = \prod_1^\infty \left\{1+\frac{z^k}{\lambda_n}\right\}^{\zeta_n}$$

$F(z)$ is canonical product[2] of order ρ. Let $r = \lambda_n^{1/k} - 1$. Then for $n \geq n_2$

$$N(r,o) < 2n\left(\lambda_n^{1/k}-1\right)\log\lambda_n.$$

$$\log M(r/R) > \zeta_n \log\left\{1+\frac{1}{\lambda_n}\frac{r^k}{R^k}\right\}$$

$$\frac{\log M(r/R)}{N(r,o)\psi(r)} > \exp\left\{\left[3\eta_n - 3\frac{1}{\theta(\lambda_n^{1/k})} - \frac{\eta r}{k}\right]\log\lambda_n\right.$$

$$\left. + \left[(4+\rho) - \left(\frac{\rho}{k}+\varepsilon_{n-1}\right) - \frac{ll\lambda_n}{\log\lambda_{n-1}}\right]\log\lambda_{n-1} + O(1)\right\}$$

$$\longrightarrow \infty \text{ With } n \longrightarrow \infty.$$

Hence (4) is proved. (5) and (3) can be proved similarly.

Finally we add that by considering $\log M(t/2)$ instead of $\log M(t)$ we can prove relations (3), (4) and (5) with $T(r)$ instead of $\log M(r)$.

(2) Cf. S. M. Shah, Bulletin American Math. Soc., 46 (1940), 909—912.

POLYTROPIC GAS SPHERES WITH VARIABLE INDEX

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SUMMARY

Eddington's problem, of how far the properties of the variable polytrope lie between those of the limiting, uniform polytropes of maximum and minimum politropic indices, has been considered. It has been shown from quite general considerations that the gravitational potential energy cannot be an isolated extremal property exhibited by the one-phase model of the limiting polytrope. Using Candler's equations, several intermediate properties have been deduced for the variable polytrope, besides those derived by Candler, in particular, the ratio of the central to the mean density. The temperature distribution in the variable polytrope has been considered in various aspects, and several integral theorems as well as the monotonic decrease from the centre to surface of certain physical variables, which is assumed for real stars, have been shown to follow from the polytropic equation.

In what follows, we will consider a sphere of perfect gas, of constant mean molecular weight, in hydrostatic equilibrium. The sphere will be called a *uniform polytrope*, if at every point in it the relation

$$P = K\rho^{1+\frac{1}{n}} \qquad \ldots \ldots \qquad (1)$$

is satisfied, where P and ρ are respectively the pressure and density at any point, and K and n are constants.

Eddington[6] showed from thermodynamic considerations that real stars approximate to the law (1) with $n=3$, and discussed in detail these configurations which have acquired the designation of "standard polytropes." Theoretical objections were raised later on against the standard model, voiced most prominently by Milne,[11] according to whom a better representation of the observed facts would be a two-phase configuration with a central polytropic core of index n_1 surrounded by a polytropic shell of different index n_2. Jeans[9] objected to the standard model on the ground that there were reasons to suspect that the perfect gas law were not obeyed right up to the central regions of the star, where the matter must be supposed to be in a liquid or semi-liquid state.

We are thus led to consider configurations in which the polytropic law (1) does not hold throughout the entire mass. In fact, all possible spherical distribution of matter can be conceived of as made up of concentric shells, the polytropic index

varying from shell to shell. Eddington[7] initiated the consideration of such models. He defined[8] the (variable) polytropic index n by the relation

$$\frac{d}{dr}(\log P) = \left(1+\frac{1}{n}\right)\frac{d}{dr}(\log \rho), \quad \ldots \quad (2)$$

where P is the pressure and ρ the density at a point distant r from the centre, and n is a function of r.

We will call a configuration satisfying the relation (2) a *variable polytrope*. When n is a constant, relation (2) reduces to (1).

Eddington proved[7] that the negative potential energy, Ω, of a variable polytrope of mass M and radius R, is given by

$$\Omega = \frac{3}{5-n_0} G\frac{M^2}{R}, \quad \ldots \quad (3)$$

where G is the gravitational constant, and n_0 is an average value of n taken in a particular manner, which, for positive values of the polytropic index, lies between its maximum and minimum values.

Thus, if the index of the variable polytrope varies between the limits n_1 and n_2, the extremal values of its potential energy are given by the uniform polytropes of the same mass and radius and indices n_1 and n_2 respectively.

In a later paper,[8] Eddington investigated the general problem of to what extent the properties of the polytrope S with variable index within the limits n_1 and n_2 lie between those of the limiting, uniform polytropes, S_1 and S_2, of respective indices n_1 and n_2. He showed from physical considerations that in actual stars the polytropic index is likely to increase monotonically from a value $n_1=1\cdot5$ at the centre to $n_2=3\cdot5$ at the surface. He showed that the potential satisfies an equation formally similar to Enden's equation. The solution curve was found to lie initially between the limiting polytropes but was liable to divergence further on. His conclusion is that the potential energy is probably the only intermediate property of the variable polytrope S, and that the extremal value of any selected property of S, subject to the condition that n increases outwards monotonically from n_1 to n_2, is given by a two-phase model with an inner portion of constant index n_1 and an outer portion of constant index n_2, the relative extent of the two portions depending on the selected property and on the characteristics (*e.g.*, mass and central density) chosen to be kept constant.

The object of the present paper is to show from quite general considerations that, contrary to Eddington's surmise, the gravitational potential energy cannot be an isolated extremal property exhibited by the one-phase model of the limiting polytrope. In fact, Candler[2] has obtained several properties, other than the potential energy, of the variable polytrope S, which lie between those of the limiting, uniform polytropes, S_1 and S_2. It should be noted that these papers have only considered

the hydrostatic equilibrium of the polytrope. Very recently, N. R. Sen[12] has considered the thermodynamic properties of a mixture of gas and radiation, and obtained an extremal property for the temperature distribution of the variable polytrope. We will consider both aspects of the case, and with the help of Candler's results[2] deduce further extremal properties of the variable polytrope, *viz.*, for the potential, temperature, mean temperature, ratio of gas to radiation pressure, and the ratio of the central to mean density. We will investigate the problem of how far correspondence exists between three gas spheres in the sense defined by Sen,[12] and consider the intermediacy of the temperature distribution by a new definition of corresponding points. We will prove Chandrasekhar's extension[5] of Eddington's theorem[6] for minimum central temperature to stars with appreciable radiation pressure by a modification of a method due to Eddington,[6] and extend the Theorem X of Chandrasekhar's paper[4] for vanishing radiation pressure. Following Chandrasekhar,[3] we will give two integral theorems on the central potential and central temperature of a perfect gas sphere of given mass and radius, and deduce two integral theorems due to Chandrasekhar[4] concerning radiation pressure and extremal property of $K=\frac{P}{\rho^{1+\frac{1}{m}}}$, where m is a constant, from the property of a variable polytrope. Certain extremal properties of Chandrasekhar's $K=\frac{P}{\rho^{1+\frac{1}{m}}}$ without Chandrasekhar's restriction of K decreasing outward[4] will be deduced from Candler's equations,[2] and a closer upper limit found for K than that given by Chandrasekhar in page 80 of his monograph,[3] leading to Chandrasekhar's Theorem 7 on the upper limit of the central radiation pressure of a given gas mass.[3] We will also show that the monotonic decrease, from the centre to surface, of certain physical variables, which is assumed for real stars, follow naturally from the hydrostatic equation and the definition of the variable polytrope.

Certain consequences, as we have just said, directly follow from the relation (2) and the equation of hydrostatic equilibrium. We state and deduce them as follows.

THEOREM I. *The pressure, density, temperature, potential and the ratio of the pressure to density at any point of a variable polytrope of positive index are monotonic decreasing functions of the distance of the point from the centre.*

That the pressure P and the potential ϕ are monotonic decreasing functions of the distance r of the point from the centre, follows from the hydrostatic equation of equilibrium, $dP=-g\,\rho dr$, which is equivalent to $dP=\rho d\phi$, where ρ is the density and g the gravitational acceleration at the point.

Now, we have from relation (2)

$$\frac{dP}{P}=\left(1+\frac{1}{n}\right)\frac{d\rho}{\rho}. \qquad (4)$$

As n is positive, we have from (4) that $d\rho$ has the same sign as dP.

Again, we have from (4)

$$d\left(\frac{P}{\rho}\right)=\frac{\rho dP-Pd\rho}{\rho^2}=\frac{Pd\rho}{n\rho^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We have from (5) that $d\left(\frac{P}{\rho}\right)$ has the same sign as $d\rho$.

Further, we have the equations of gas and radiation pressure

$$\frac{\beta P}{\rho}=\frac{k}{\mu H}T, \text{ and } (1-\beta)\ P=\frac{1}{3}aT^4, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where k is the Boltzmann constant, μ the mean molecular weight, H the mass of the proton, and a the Stefan-Boltzmann constant.

We have from (6), by differentiation,

$$\beta d\left(\frac{P}{\rho}\right)+\frac{P}{\rho}\,d\beta=\frac{k}{\mu H}dT \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and
$$(1-\beta)\,dP-Pd\beta=\tfrac{4}{3}aT^3\,dT. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It is easily seen from (7) and (8) that dT is negative for all values of $d\beta$, positive, zero or negative.

Ritter, Eddington, Milne and Chandrasekhar[3] have derived a number of integral theorems on the hydrostatic equilibrium of a gas sphere. Chandrasekhar[3] has very succinctly summed up the main results in the statement that, subject to the condition that the mean density $\bar{\rho}(r)$ inside a sphere of radius r does not increase outwards, the physical variables characterising the given equilibrium configuration of central density ρ_c and mean density $\bar{\rho}$, namely, the central pressure, potential energy, mean values, with respect to the mass, of the gravity, pressure and temperature (for the case of negligible radiation pressure), have values respectively less than those for the configuration of uniform density with $\rho=\rho_c$, and respectively greater than those for the configuration of uniform density with $\rho=\bar{\rho}$, the mass remaining the same. We can, adopting Chandrasekhar's method, prove the same property for the central potential, ϕ_c, as follows.

THEOREM II. *The central potential ϕ_c of any equilibrium configuration of mass M and radius R, in which the mean density $\bar{\rho}(r)$ inside r does not increase outward, must satisfy the inequality*

$$\frac{3}{2}\,G\frac{M}{R}\leq\phi_c\leq\frac{3}{2}\,G\frac{M}{R_c},$$

where R and R_c are the radii of spheres of mass M and uniform densities $\bar{\rho}$ and ρ_c respectively.

where R_c is the radius of the sphere of the given mass M and uniform density equal to the given central density ρ_c.

We can easily see in a general way the truth of Chandrasekhar's summary[3] of the integral theorems.

The physical variables characterising a given equilibrium configuration can be put into the following integral forms[3] :

$$\left.\begin{aligned} P_c &= \frac{G}{4\pi}\int\frac{M(r)\,dM(r)}{r^4}, \quad \Omega = G\int\frac{M(r)\,dM(r)}{r}, \\ M\bar{g} &= G\int\frac{M(r)\,dM(r)}{r^2}, \quad M\bar{T} = \frac{G}{3}\,\frac{\mu H}{k}\int\frac{M(r)\,dM(r)}{r}, \\ M\bar{P} &= \frac{G}{4\pi}\int\frac{M^2(r)\,dM(r)}{r^4} \text{ and } \phi_c = G\int\frac{dM(r)}{r}, \end{aligned}\right\} \quad . \quad . \quad (14)$$

where $\bar{g}$, $\bar{P}$ and $\bar{T}$ are the mean values, with respect to the mass, of the gravity, pressure and temperature respectively, Ω is the negative potential energy, and the integrals are taken from the centre to the surface.

It will be clear from (14) that the physical variables can all be represented by the general integral

$$I_{\sigma,\nu} = \int_0^M \frac{M^\sigma(r)\,d\,M(r)}{r^\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where σ and ν have integral (including zero) values.

We have from (15)

$$I_{\sigma,\nu} = \frac{1}{\sigma+1}\int_0^M \frac{d\,[M^{\sigma+1}(r)]}{r^\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The integral (16) is a minimum when, for each step of $M^{\sigma+1}(r)$, r is as great as possible. This, under the condition that $\bar{\rho}(r)$ is not to increase outwards, will lead to the sphere of the same mass and of uniform density equal to the mean density $\bar{\rho}$. That this is so can be shown by dividing the sphere into shells of equal mass. The maximum value is similarly seen to be that of the sphere of the same mass and of uniform density equal to the central density ρ_c.

It may be remarked that Eddington[6] has obtained minimal theorems subject to the condition that the density ρ at any point does not increase outwards. We can easily see as follows that Chandrasekhar's condition, that the mean density $\bar{\rho}(r)$ inside a sphere of radius r does not increase outwards, is less stringent than Eddington's.

We have

$$\frac{4\pi}{3}\bar{\rho}(r) = \frac{M(r)}{r^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Differentiating (17) with respect to r, we have

$$\frac{4\pi}{3}\frac{d\bar{\rho}(r)}{dr} = \frac{r^3\dfrac{dM(r)}{dr} - 3r^2\ M(r)}{r^6}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Substituting $dM(r) = 4\pi r^2\rho\, dr$, we have from (18).

$$\frac{d\bar{\rho}(r)}{dr} = \frac{3}{r}[\rho - \bar{\rho}(r)], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We have from (19) that the condition $\frac{d\bar{\rho}(r)}{dr} \leq 0$ is equivalent to $\rho \leq \bar{\rho}(r)$.

Thus, at a point r where the density is ρ and mean density $\bar{\rho}(r)$, where $\rho \leq \bar{\rho}(r)$, we can introduce a thin, spherical shell of matter of any density between ρ and $\bar{\rho}(r)$, thereby reversing Eddington's density gradient, but not Chandrasekhar's.

It is at once evident from Chandrasekhar's summary[4] of his integral theorems that most of the physical variables characterising an equilibrium configuration with polytropic index varying between the values O and n do attain extremal values in the *one*-phase model of the limiting, uniform polytrope. As Chandrasekhar's theorems are independent of the mode of variation of the polytropic index, the *minimum* value for given mass and radius is doubtless attained in the *one*-phase model of zero polytropic index, that is, the uniform density sphere of the same mass and radius. This is borne out by Candler's[2] Theorem II, which gives minimum central pressure and density for polytropes of minimum index. If, instead of the mass and radius, we keep the mass and central density the same for all configurations, the *maximum* value is now attained in the *one*-phase model of zero polytropic index, that is, the uniform sphere of the given mass and density equal to the given central density.

However, in view of Eddington's general investigation,[8] we cannot maintain that the one-phase models of limiting indices would give the extremal values of all the physical variables. Eddington[6] has, in fact, obtained a two-phase model for the minimum central temperature, namely, an isothermal sphere surrounded by a region of uniform density, which gives a range to the polytropic index from zero to infinity.

Candler[2] has reduced the Emden equation[1] to a first-order one by introducing the following dimensionless variables X and Y which are invariants of the Emden equation involving the Emden function and its first derivative, with respect to a Lane transformation :

$$X = \left(\frac{2\pi G}{3}\right)^{\frac{1}{2}} \frac{r\bar{\rho}(r)}{P^{\frac{1}{2}}}, \quad Y = \left(\frac{2\pi G}{3}\right)^{\frac{1}{2}} \frac{r\rho}{P^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In fact, as Sen[13] has remarked, if we transform X and Y to usual polytropic variables for n = constant, we obtain

$$Y = \sqrt{\frac{n+1}{6}}\, I_1, \quad X = \sqrt{\frac{3(n+1)}{2}} \cdot \frac{I_2}{I_1^{\frac{n+1}{n-1}}}, \qquad (21)$$

where I_1 and I_2 are the two invariants

$$I_1 = \xi u^{\frac{n-1}{2}}, \quad I_2 = -\xi^{\frac{n+1}{n-1}} \frac{du}{d\xi}, \qquad (22)$$

ξ and u being the variables of the normalised Emden equation.[1]

Sen[14] has analytically proved that I_1, I_2, each has a single maximum for given n, which throws light on the general nature of Candler's curves.

Candler's variable X is a positive quantity increasing from zero at the centre to infinity at the surface; the variable Y is a positive quantity increasing initially from zero at the centre, and reaching a value at the surface which may be zero, finite or infinite according to the surface value of n.

Candler defines *corresponding points* for different configurations as those with the same value of X, and gives the following reason for his choice of the variables:

"The choice of X as the variable which determines what points in the two stars are supposed to correspond is arbitrary, and similar results could be obtained by a different choice. It has the advantage of simplicity, however, in that X has the same range of values, from zero to infinity for any star, and the analysis is the same whether we are comparing two stars with the same central pressure and density or with the same mass and radius. Since we are ultimately concerned with the comparison of integral properties of the two stars, the choice is not of great importance."

With his variables X and Y Candler derives the following first-order equation:

$$\frac{dY}{dX} = \frac{Y - \frac{n-1}{n+1} XY^2}{(3+X^2)Y - 2X}, \qquad (23)$$

He gives explicit solutions of (23) for $n=0$, 1 and 5, and draws the solution-curves for the polytropes $n=0$, 1, 3 and 5.

The following properties of the solution-curves as deduced by Candler from his fundamental differential equation (23) are relevant to our paper:

(i) All the solution-curves leave the origin with unit slope.

(ii) If the polytropic indices n, n^1 of two configurations S, S^1 are such that $-1<n<n^1\leq 5$ at all "corresponding" points (that is, points with the same value of X), then the solution-curve for S lies entirely above the solution-curve for S^1.

It is property (ii) of the solution-curves that gives them their importance. Candler has expressed the physical variables in terms of X and Y and with the help of property (ii) shown several characteristics to lie between those of the limiting, uniform polytropes for two classes of variable polytropes, *viz*, (*a*) of fixed mass and radius and (*b*) of fixed central pressure and density. In particular, he has shown the intermediacy of the central pressure and density for class (*a*).

We will now use Candler's results to find further intermediate properties of the variable polytrope. We suppose, as we have already stated in the beginning of our paper, all polytropes to be perfect gas spheres of constant mean molecular weight μ.

It should be noted that Candler's theorems hold only for those characteristics which vary monotonically with the index for polytropes of uniform but different polytropic indices, as can be seen from the British Association Tables of Emden functions.[1] The reason for this is apparent. The same is particularly true[1] of the ratio of the central to the mean density, which is *purely* a function of n and assumes large values for $n>3$. We prove the intermediacy property for this ratio as follows.

THEOREM IV. *If the polytropic indices n, n^1 of two polytropes S, S^1 vary in any manner between the centre and surface such that $-1<n<n^1\leq 5$ at all corresponding points, then the ratio $\rho_c/\bar{\rho}$ of the central to the mean density is less for S than for S^1.*

Denoting the variables of the star S^1 by dashes, we have, from Candler's equation (24)

$$\frac{\rho_c/\bar{\rho}}{\rho^1_c/\bar{\rho}^1} = \exp 3\int_0^\infty \frac{(Y^1-Y)(1+X^2)dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y^1-2X\}}, \qquad (24)$$

Candler has shown that the denominator of the integrand is positive and $Y>Y^1$ for $n<n^1$. Hence the integral is negative and the ratio (24) <1. The theorem follows.

COR *If the polytropic index n of a polytrope S varies in any manner between a lower limit $n_1(>-1)$ and an upper limit n_2 (≤ 5), then the ratio of the central to the mean density of S is greater than that of S_1 and less than that of S_2, where S_1 and S_2 are uniform polytropes with the respective indices n_1 and n_2.*

It should be noted that the theorem is perfectly general, concerning only the polytropic index and independent of the physical variables characterising the star, that is, it is true for all stars, whether of the same mass, radius, etc, or not. Further, the polytropic index n need not be subject to Eddington's restriction[8] of

monotonic increase from the centre outwards, but may vary in any manner within limits $n_1(>-1)$ and $n_2(\leq 5)$. This disproves the following remark of Eddington[8]:

"But, so far as we know, the gravitational potential energy is the only quantity to which this* applies; and there are certainly some quantities, *e.g.*, the ratio of the central to the mean density, which do not always lie between the values given by the two limiting polytropes."

We next consider the potential ϕ of the variable polytrope.

THEOREM V. *The potential ϕ is less for S than for S^1 at corresponding points (that is, points with the some value of X), where S and S^1 are two polytropes with the same mass and radius, whose polytropic indices n and n^1 vary in any manner between the centre and surface such that $-1<n<n^1\leq 5$ at all corresponding points.*

Taking the zero of the potential on the surface of the star, we have

$$\phi = \int_r^R g(r)\, dr, \qquad (25)$$

Let r, r^1 and $g(r)$, $g^1(r^1)$ be the values respectively of the distance from the centre and the gravity at corresponding points. Then $g(r)$, $g^1(r^1)$ are both positive. Also Candler[2] has shown that $r>r^1$ and $g(r)<g^1(r^1)$. Thus we have

$$\phi = \int_r^R g(r)dr < \phi^1 = \int_{r^1}^R g^1(r^1)\, dr^1, \qquad (26)$$

as the correspondence is one to one, and the length of the interval in the first integral is less than that in the second.

COR. 1. *Under the conditions of Theo. V, the central potential ϕ_c of S is less than that of S^1.*

COR. 2. *If the polytropic index n of a polytrope S varies in any manner between a lower limit n_1 (>-1) and an upper limit n_2 (≤ 5), then the potential at corresponding points is greater for S than for S_1 and less for S than for S_2, where S_1 and S_2 are the uniform polytropes with the respective indices n_1 and n_2 and having the same mass and radius as S.*

COR. 3. *The negative potential energy Ω of the variable polytrope S of Cor. 2 lies between its values for the limiting polytropes S_1 and S_2.*

As
$$\Omega = \tfrac{1}{2}\int \phi\, dm, \qquad (27)$$

*The property of intermediacy.

where dm is an element of mass at the point where the potential is ϕ, the corollary follows from (27) on constructing shells of equal mass at corresponding points.

Eddington[7] has derived this property of Ω by a different method.

We can similarly prove the following theorem for stars of constant central pressure and density.

THEOREM VI. *If the polytropic index n of a polytrope S varies in any manner between a lower limit* $n_1 (> -1)$ *and an upper limit* n_2 (≤ 5), *then the potential at corresponding points is less for S than for* S_1 *and greater for S than for* S_2, *where* S_1 *and* S_2 *are the uniform polytropes with the respective indices* n_1 *and* n_2 *and having the same central pressure and density as S.*

We will now consider the mean temperature $\overline{\mathrm{T}}$ with respect to the mass, and β, the ratio of gas to total pressure.

THEOREM VII. *If the polytropic index n of a polytrope S varies in any manner between a lower limit* n_1 (>0) *and an upper limit* n_2 (≤ 5), *then the mean temperature*

$$\overline{T}(r) = \frac{1}{M(r)}\int_0^r Td\,M(r)$$

is greater than that of S_1 *and less than that of* S_2 *at corresponding points, so long as* $X \geq \sqrt{3}$, *where* S_1 *and* S_2 *are the uniform polytropes, with respective indices* n_1 *and* n_2, *having the same mass and radius as S. (Radiation pressure is neglected.)*

We have

$$d\,\mathrm{M}(r) = 4\pi r^2 \rho dr, \qquad (28)$$

where $\mathrm{M}(r)$ is the mass inside the sphere of radius r,

From (28) and the equation (6) of gas pressure, we have

$$\int_0^r \mathrm{T}d\mathrm{M}(r) = \int_0^r \frac{\mu\mathrm{H}}{k}\frac{\mathrm{P}}{\rho}\,d\,\mathrm{M}(r) = \frac{4\pi\mu\mathrm{H}}{k}\int_0^r \mathrm{P}r^2\,dr, \qquad (29)$$

Now, by a theorem due to Milne,[10] we have

$$\Omega(r) = 12\pi\int_0^r \mathrm{P}r^2 dr, \qquad (30)$$

where $\Omega(r)$ is the negative potential energy of the sphere of radius r.

From (29) and (30), we have

$$\overline{T}(r) = \frac{\mu H}{3k} \cdot \frac{\Omega(r)}{M(r)}$$

$$= \frac{1}{5-n_0} \cdot \frac{G\mu H}{k} \cdot \frac{M(r)}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

from equation (3) due to Eddington.[7]

Denoting the corresponding quantities for the star S_2 by dashes, we have, similarly

$$\overline{T}''(r) = \frac{1}{5-n_2} \cdot \frac{G\mu H}{k} \cdot \frac{M(r)}{r''}, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Now, we have, according to equations (47) and (48) in Candler's paper,[2]

$$\frac{M(r)/M''(r)}{r/r''} = \exp \int_X^\infty \frac{(3-X^2)(Y-Y')\,dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y'-2X\}} \cdot \quad . \quad (33)$$

Candler has proved the denominator of the integrand to be positive and $Y>Y'$ for $n<n'$. Also $X \geq \sqrt{3}$, by hypothesis. Hence the integral in (33) is negative, and the ratio (33) <1.

Eddington[7] has proved, further, that $n_0 < n_2$. Hence, we have

$$\overline{T}(r) < \overline{T}''(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The other part of the theorem follows in like manner.

Cor. *Under the conditions of Theo VII, the mean temperature of a variable polytrope S is greater than that of S_1 and less than that of S_2, where S_1 and S_2 are the uniform, limiting polytropes of the same mass and radius as S.*

We have the following two theorems on the intermediacy of β, the ratio of gas to total pressure.

Theorem VIII. *If the polytropic indices n, n^1, of two polytropes, S, S^1, of the same mass and radius, vary in any manner between the centre and surface such that $-1<n<n^1\leq 5$ at all corresponding points, then the value of β, the ratio of gas to total pressure, is less at any point in S than at the corresponding point in S^1, so long as X is small, and the converse when X is large.*

Eliminating T between the equations (6) for gas and radiation pressure, we have

$$\frac{1-\beta}{\beta^4} = \frac{a}{3}\left(\frac{\mu H}{k}\right)^4 \frac{P^3}{\rho^4} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

From Candler's[2] equations (25^1) and (46), we have

$$\frac{P^3/\rho^4}{(P^1)^3/(\rho^1)^4} = \left(\frac{Y^1}{Y}\right)^4 \exp \int_X^{\infty} \frac{12\,(Y-Y^1)\,dX}{\{(3+X^2)\,Y-2X\}\,\{(3+X^2)\,Y^1-2X\}} \quad . \quad (36)$$

Now, from the property (i) of Candler's curves,[2] $\frac{Y^1}{Y}\to 1$ as $X\to 0$. Also the integral in (36)$\to 0$, as $X\to\infty$. Hence, we have, from (35) and (36),

$\beta<\beta^1$ when X is small,

and $\beta>\beta^1$, for large X.

COR. 1. *Under the conditions of Theo. VIII, the central and the surface values of* β, *the ratio of gas to total pressure, for a variable polytrope S lie between their respective values for the limiting, uniform polytropes* S_1 *and* S_2, *of the same mass and radius as S.*

COR. 2. *There will be one or more points at which the polytropes S,* S_1 *and* S_2 *of Theo. VIII will have the same value of* β.

This follows, as β varies continuously with r, the distance of the point from the centre.

It should be noted that for polytropes of positive indices, the minimum value of βc, according to Theo. VIII, Cor. 1, is attained in the uniform density sphere ($n=0$). This leads to, as we will immediately explain, the following theorem due to Chandrasekhar[3] :

The ratio ($1-\beta_c$) *of the radiation pressure to the total pressure at the centre of a wholly gaseous configuration in equilibrium in which* $\bar{\rho}\,(r)$ *does not increase outward, satisfies the inequality*

$$1-\beta_c \leq 1-\beta^*, \quad . \quad . \quad . \quad . \quad (37)$$

where β^* *satisfies the quartic equation*

$$M = \left(\frac{6}{\pi}\right)^{\frac{1}{2}}\left[\left(\frac{k}{\mu_c H}\right)^4 \frac{3}{a}\,\frac{1-\beta^*}{\beta^{*4}}\right]^{\frac{1}{2}} \frac{1}{G^{\frac{3}{2}}}\,; \quad . \quad . \quad (38)$$

μ_c *is the mean molecular weight at the centre.*

The β^* of (38) is the β_c for a uniform density star. This follows from equation (35) and the expression[6] for P_c the central pressure, of a uniform density star of mass M and radius R, *viz.*,

$$P_c = \frac{3}{8\pi}\,\frac{GM^2}{R^4}. \quad . \quad . \quad . \quad . \quad (39)$$

THEOREM IX. *If the polytropic indices* n, n^1, *of two polytropes* S, S^1, *with the same central pressure and density, vary in any manner between the centre and surface such that* $-1<n<n^1\leq 5$ *at all corresponding points, then the value of* β, *the*

ratio of gas to total pressure, is greater at any point (except the centre) in S than at the corresponding point in S^1.

We have, from Candler's[2] equations (42) and (43),

$$\frac{P^3/\rho^4}{(P^1)^3/(\rho^1)^4} = \left(\frac{Y^1}{Y}\right)^4 \exp \int_0^X \frac{12\,(Y^1-Y)dX}{\{(3+X^2)\,Y-2X\}\,\{(3+X^2)\,Y^1-2X\}} \quad . \tag{40}$$

From (35) and (40), the theorem follows for all $X \neq 0$.

COR. *If the polytropic index n of a polytrope S varies in any manner between a lower limit n_1 (>-1) and an upper limit n_2 (≤ 5), then the value of β, the ratio of gas to total pressure, at corresponding points (except the centre), is less for S than for S_1 and greater for S than for S_2, where S_1 and S_2 are the uniform polytropes, with respective indices n_1 and n_2, and the same central pressure and density as S.*

From equations (6) and (35) it follows that β_c and T_c are the same for the polytropes S, S_1 and S_2 of Theo. IX. These are the configurations which have been the subject of a thermodynamic investigation by Sen in a recent paper[12] in which he has obtained the following extremal theorem on the temperature distribution of a variable polytrope:

If $n<3$ (or >3), the temperature of the given configuration at any point is intermediate between the temperatures at corresponding points of the polytropes n_1 and n_2 with the same central density and temperature as the given mass.

Sen defines corresponding points as those with the same value of β and remarks, "Any two such masses need not have corresponding points, but for the purpose of the arguments which follow, the different gas masses will be supposed to have corresponding points."

It follows from Theo. IX, Cor., that Sen's configurations are intermediate with respect to β at Candler's corresponding points (*i.e.*, points with the same value of X). Hence, if β vary monotonically from the centre to surface for each of the three configurations S, S_1 and S_2, we shall have a continuum of unique (one to one) corresponding points, starting from the centre, which will not, however, extend to the surface of the variable polytrope S. With the help of the following theorem, we will investigate in greater detail the existence of corresponding points as defined by Sen.

THEOREM X. *For any mode of variation of the polytropic index n, β increases or decreases monotonically from the centre to surface, according as $n \lessgtr 3$; β is constant only for the standard polytrope ($n=3$).*

From the equations (6) for gas and radiation pressure, we have

$$\frac{1-\beta}{\beta}=\frac{a}{3}\,\frac{\mu H}{k}\,\frac{T^3}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (41)$$

Hence, we have

$$d\left(\frac{1-\beta}{\beta}\right)=\frac{a}{3}\,\frac{\mu H}{k}\,\frac{3T^2\rho dT-T^3d\rho}{\rho^2}$$

$$=\frac{a}{3}\,\frac{\mu H}{k}\,\frac{T^2(3-n^1)}{\rho}\,dT, \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

where

$$n^1=\frac{d\,(\log\rho)}{d\,(\log T)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

is the polytropic index for gas pressure alone.[12]

Now, Sen[12] has shown shat

$$n^1 \gtreqless 3 : \ : n \gtreqless 3 .$$

Hence our theorem follows, as we have shown in Theo. I that the temperature decreases outwards.

It may be noticed that Chandrasekhar[4] has obtained the necessary and sufficient condition for $(1-\beta)$ decreasing outward to be that $k\eta$ at any point inside the star must be greater than the average value of $k\eta$ for material exterior to r, which is equivalent to $\overline{k\eta(r)}$ decreasing outward, where $\overline{k\eta(r)}$ is the mean value of $k\eta(r)$ with respect to the pressure, k being the opacity co-efficient and $\eta(r)=\dfrac{L(r)/M(r)}{L/M}$, $L(r)$ denoting the luminosity for mass $M(r)$. We have shown in Theo. X that for a variable polytrope the necessary and sufficient condition for $(1-\beta)$ decreasing outward can be stated in the simple form that the polytropic index n should nowhere exceed the value 3.

The following results regarding the existence of corresponding points in Sen's configurations[12] follow from Theos. IX and X.

Case I. $n_1<n_2<3$.

There is a continuum of unique (one to one) corresponding points, starting from the centre, but not extending to the surface, of S. There is correspondence, however, between S and S_1, right up to the surface of S.

Case II. $n_2>n_1>3$.

There is again a continuum of unique, corresponding points starting from the centre, which does not extend to the surface of S. There is correspondence, however, between S and S_2, right up to the surface of S.

CASE III. $n_1=3, n_2>3$.

There is only one corresponding point for the three polytropes, *viz.*, the centre. There is correspondence, however, between S and S_2, right up to the surface of S.

CASE IV. $n_1<3, n_2=3$.

The centre is the only corresponding point for the three polytropes. There is correspondence, however, between S and S_1, right up to the surface of S

CASE V. $n_1<3, n_2>3$.

The centre is the only corresponding point for the three polytropes. There will not be *unique* correspondence even between two polytropes, unless the variable polytrope S consist of two regions, *viz.*, a central core with $n<3$ (or >3) and an envelope with $n>3$ (or <3).

We will first consider the case of a central core in which $n<3$ with an envelope where $n>3$. β will at first increase in S and then decrease. In the interval at the end of which β returns to its initial central value (β_c), two points in S will correspond to one in S_1. Afterwards the correspondence will be unique between S and S_2 right up to the surface of S.

Similar considerations will apply to the reverse model.

We thus arrive at the conclusion that for the existence of corresponding points, as defined by Sen,[1,2] *for all the three configurations*, the index of the variable polytrope must vary between limits which are *both* less (or greater) than 3, and in these cases a continuum of *unique* (one to one) corresponding points exists, beginning from the centre, but not extending right up to the surface, of the variable polytrope.

Neglecting radiation pressure, we will derive the following theorem on the intermediacy of the temperature distribution in a variable polytrope.

THEOREM XI. *If the polytropic index n of a polytrope S varies in any manner between a lower limit $n_1(>-1)$ and an upper limit n_2 (≤ 5), then the temperature at corresponding points, so long as $0<X\leq\sqrt{3}$, is greater for S than for S_1 and less for S than for S_2, where S_1 and S_2 are the uniform polytropes, with respective indices n_1 and n_2, and the same central pressure and density as S.* (*Radiation pressure is neglected.*)

We have, from the gas equation (6),

$$\frac{T}{T''}=\frac{P/\rho}{P''/\rho''}, \qquad (44)$$

where the dashes refer to S_2.

From (44) and Candler's[2] equations (42) and (43), we have

$$\frac{T}{T''}=\frac{Y''}{Y}\exp\int_0^X\frac{(X^2-3)(Y-Y'')\,dX}{\{(3+X^2)Y-2X\}\{(3+X^2)Y''-2X\}} \qquad (45)$$

The ratio (45) <1 for $0<X\leq\sqrt{3}$, and the theorem follows

We have thus proved the intermediacy of the temperature distribution for polytropes of constant central pressure and density and negligible radiation pressure, for a small region surrounding the centre (except of course the centre itself). We can similarly prove the following theorem on the intermediacy of the temperature distribution, for all parts except a central core, in polytropes of the same mass and radius.

THEOREM XII. *If the polytropic index n of a polytrope S varies in any manner between a lower limit n_1 (> -1) and an upper limit n_2 ($\leqq 5$), then the temperature at corresponding points, so long as $X \geq \sqrt{3}$, is greater for S than for S_1 and less for S than for S_2, where S_1 and S_2 are the uniform polytropes, with respective indices n_1 and n_2, and the same mass and radius as S.*

It may be noted that we cannot extend the theorem so as to include the centre of the polytrope, as the central temperature is not a monotonic function[1] of n, and the one-phase model cannot, therefore, give the minimum central temperature. In fact, as we have already remarked, Eddington[6] has obtained a two-phase model for the minimum central temperature.

It thus appears that neither the fixing of the mass and radius nor that of the central pressure and density will ensure the intermediacy of the temperature distribution throughout the polytrope. By a suitable definition of corresponding points, we arrive at the following model, which has a similar intermediate property for the temperature distribution.

We have proved in Theorem I that the potential at any point decreases monotonically from the centre outwards. Polytropes of the same mass M and radius R have the same surface potential $G\frac{M}{R}$. We define corresponding points as those where the potential is the same. We note that the correspondence so defined is unique, and we have a continuum of corresponding points, starting from the boundary, for polytropes of the same mass and radius.

Now, Eddington[7] has shown that

$$\frac{P}{\rho} = \frac{1}{\bar{n}+1}\,\phi, \qquad (46)$$

where $\bar{n}$ is an average value of n lying between its extreme values (assumed positive).

From (46) and the gas equation (6), we immediately have

$$T = \frac{\mu H}{k}\,\frac{\phi}{\bar{n}+1}, \qquad (47)$$

neglecting radiation pressure.

As $\bar{n}$ lies between the limiting indicial values, we have proved the following theorem:

THEOREM XIII. *If S be a polytrope whose index varies in any manner between the positive values n_1 and n_2, and S_1 and S_2 be the uniform polytropes*

with the respective indices n_1 and n_2 and the same mass and radius as S, the temperature at any point in S lies between those at the corresponding (of equal potential) points in S_1 and S_2. (Radiation pressure is neglected.)

We remark that when $n_2 \leq 5$, the central potential of S lies between those of S_1 and S_2, by Theorem V, Cor. 2; hence the correspondence does not extend right up to the centre of S, and we cannot assert the intermediacy of the central temperature. There will be complete correspondence, however, between S and S_2, and the central temperature of S will be greater than the temperature at the corresponding point (which will not be the centre) of S_2.

It may be noted that most of the integral theorems dealing with the temperature neglect the radiation pressure. Chandrasekhar, in a recent paper,[5] has generalised Eddington's minimal theorem[6] for central temperature, taking radiation pressure into account, as follows :

In any gaseous equilibrium configuration of prescribed mass and radius and of constant mean molecular weight μ, in which both ρ and T do not increase outward, the minimum value of T_c is attained in the sequence of equilibrium configurations which consist of isothermal cores and homogeneous envelopes.

By a study of the appropriate sequence of composite configurations, the minimum central temperature has been found to assume the form

$$(T_c)_{min.} = \tfrac{1}{2} Q_{min.} \frac{\mu H}{k} \frac{GM}{R}, \qquad \ldots \ldots \qquad (48)$$

where $Q_{min.}$ is a dimensionless quantity depending on $M\mu^2$ only. Chandrasekhar has tabulated $Q_{min.}$ as a function of the mass of the configuration expressed in a suitable unit. It has been found that $Q_{min.}$ monotonically decreases with increasing mass from the value 0·640 ($M=0$) to 0 ($M \to \infty$).

In the course of his argument, Chandrasekhar has proved the following general theorem :

If $K(P, \rho)$ be an arbitrary continuous function in the variables P and ρ such that

$$2P \frac{\partial K}{\partial P} + \rho \frac{\partial K}{\partial \rho} > 0 \quad and \quad \frac{\partial K}{\partial P} > 0,$$

then in any equilibrium configuration in which both $K(P, \rho)$ and ρ do not increase outward, the minimum value of $K_c \equiv K(P_c, \rho_c)$ is attained in the sequence of equilibrium configurations which consist of cores of constant K and homogeneous envelopes.

We will prove a similar theorem under less stringent conditions.

THEOREM XIV. *If $K(P, \rho)$ be an arbitrary continuous function in the variables P and ρ such that $\frac{\partial K}{\partial P} > 0$ and not decreasing from the centre to surface, and $\frac{\partial K}{\partial \rho} < 0$, then in any equilibrium configuration in which both ρ and $K(P, \rho)$ do not increase outward, the minimum value of $K_c \equiv K(P_c, \rho_c)$ is attained in the sequence of equilibrium configurations which consist of cores of constant K and homogeneous envelopes.*

Our proof is a modification of the argument by which Eddington has proved his minimal theorem[6] for the central temperature of a star of negligible radiation pressure.

We will first prove the following Lemma.

LEMMA. *In the configuration in which K_c attains the minimum, ρ and K cannot simultaneously be monotonically decreasing at any point r in the interval $0 \leq r \leq R$.*

For, if not, consider three consecutive infinitesimal spherical shells with $\rho_1 > \rho_2 > \rho_3$ and $K_1 > K_2 > K_3$. Since $\rho_2 > \rho_3$, we can remove a small mass from the second to the third shell without reversing the density gradient It can be easily seen that this will reduce the pressure interior to the shell ρ_2 by a constant amount. It should be noted that this constant reduction of pressure will not affect the hydrostatic equilibrium interior to ρ_2, as the hydrostatic equation remains unaltered. The change in K interior to the shell ρ_2 will be $\frac{\partial K}{\partial P} dP$, and as we have supposed $\frac{\partial K}{\partial P} > 0$ and not decreasing from the centre to surface, there will be a reduction in K, least (but not zero) at the centre, and increasing (or stationary) outwards. Thus the gradient of K inside ρ_2 is not reversed, and outside ρ_3 it remains as before. Since $K_1 > K_2 > K_3$, there is a small margin for changes in K in the two shells without reversing the gradient of K.

Thus, we can always reduce K_c without violating the conditions by making a small transfer of mass at a point where both $\rho_1 > \rho_2 > \rho_3$ and $K_1 > K_2 > K_3$. Hence, for minimum K_c, both these inequalities cannot be *simultaneously* true. It follows that in any part or whole of the interval $0 \leq r \leq R$, ρ and K cannot be *simultaneously* monotonically decreasing. The configuration of minimum K_c must therefore be capable of division into spherical shells in which ρ and K are alternately constant. ρ and K cannot both be constant in the same spherical shell, as this would make P constant in that shell and ρ would consequently be zero from the equation of hydrostatic equilibrium.

It should be noted that both Eddington's[6] and Chandrasekhar's[5] enunciation of the Lemma proved here (*mutatis mutandis*) are slightly incorrect, as, at any

point in the boundary between two shells of constant ρ and K, both $\frac{d\rho}{dr}$ and $\frac{dK}{dr}$ are non-existent.

Now, the central part of the configuration with minimum K_c must have constant K, that is, be a polytrope of index m. Otherwise, if K be not constant in a central core, take a mass δM from the outer part of the configuration and distribute it uniformly through a small sphere at the centre, of radius r and finite density ρ. Make r tend to zero, keeping ρ finite. From equation (39), we have the central pressure for the uniform sphere $=\frac{2\pi}{3}Gr^2\rho^2$. Hence the increase of central pressure tends to zero as r tends to zero, while ρ, which is the increase in the central density, remains finite. Thus, if r be taken small enough, K_c will decrease finitely, as $\frac{\partial K}{\partial \rho}<0$. This transfer is ruled out if the configuration have a polytropic central core for which K is constant, as it would lead to a reversal of the K gradient.

Our argument here differs somewhat from that of Eddington[6] who has made the increase of central pressure tend to infinity. It may be pointed out that this can be so only if δM be kept finite. In that case, it follows from equation (39) that the increase of central pressure $\propto\frac{(\delta M)^2}{r^4}$, while the increase of central density $\propto\frac{\delta M}{r^3}$. Consequently, the central pressure increases infinitely faster than the central density, and the central temperature will *increase*, if r be made sufficiently small. (In fact, the increase of central temperature $\rightarrow\infty$, as $r\rightarrow 0$.)

The rest of the configuration must be uniform, for any polytropic shell interposed in between would, by greater concentration of mass towards the centre, increase P_c and hence K_c $\left(\text{as } \frac{\partial K}{\partial P}>0\right)$.

We note that the proofs given by Eddington[6] and Chandrasekhar[5] of their theorems are slightly incomplete. Eddington has not proved that there can be no isothermal shell in the uniform part of the star, and Chandrasekhar has not proved that the central part of the star must be polytropic.

Now the distribution of the temperature T in a polytrope is a continuous function of P and ρ and satisfies the conditions of Theo. XIV.

This we can see as follows.

From the equations (6) we have

$$P=\frac{k}{\mu H}\rho T+\tfrac{1}{3}aT^4. \qquad (48)$$

From (48), by differentiation, we have

$$\frac{\partial T}{\partial P} = \frac{1}{\frac{k}{\mu H}\rho + \frac{4}{3}aT^3} \quad \text{and} \quad \frac{\partial T}{\partial \rho} = -\frac{T}{\frac{k}{\mu H}\rho + \frac{4}{3}aT^3}. \qquad (49)$$

As ρ and T are positive and do not increase outwards, (49) shows that the conditions of Theorem XIV are satisfied.

This leads to Chandrasekhar's minimal theorem[5] on the central temperature of a gaseous configuration, taking radiation pressure into account.

If we take

$$K = \frac{P}{\rho^{1+\frac{1}{m}}}, \qquad (50)$$

where m is a positive constant, we have

$$\frac{\partial K}{\partial P} = \frac{1}{\rho^{1+\frac{1}{m}}} \quad \text{and} \quad \frac{\partial K}{\partial \rho} = -\left(1 + \frac{1}{m}\right)\frac{P}{\rho^{2+\frac{1}{m}}}. \qquad (51)$$

Hence K given by (50) satisfies the conditions of Theorem XIV, and we have the following Corollary to Theorem XIV.

THEOREM XIV, COR. *In any equilibrium configuration of prescribed mass and radius in which both* ρ *and* $K = \frac{P}{\rho^{1+\frac{1}{m}}}$ ($m > 0$) *do not increase outward, the minimum value of* K_c *is attained in the sequence of equilibrium configurations which consist of polytropic cores of index* m *and homogeneous envelopes.*

It may be noted that our corollary is an extension to all positive values of m of a theorem due to Chandrasekhar.[4]

The following theorem gives the necessary and sufficient condition for K defined by (50) not to increase outward.

THEOREM XV. *The necessary and sufficient condition that* $K = \frac{P}{\rho^{1+\frac{1}{m}}}$ ($m > 0$) *does not increase as we proceed outward from the centre to the surface of a variable polytrope is that* $m \geq$ *the maximum index in the polytrope.*

Logarithmically differentiating (50), we have

$$\frac{dK}{K} = \frac{dP}{P} - \left(1 + \frac{1}{m}\right)\frac{d\rho}{\rho}. \qquad (52)$$

From (2) and (52) we have

$$\frac{dK}{K} = \left(\frac{1}{n} - \frac{1}{m}\right)\frac{d\rho}{\rho}, \qquad (53)$$

where n is the polytropic index at the point considered, and our theorem follows from (53).

Now, for $0<m \leqq 3$, Candler[2] has proved that K_c decreases with increasing polytropic index n. Hence K_c is minimum for the uniform polytrope of maximum polytropic index which must be m. We have thus proved the following extension of the Theorem 11 in Chandrasekhar's paper[4] :

THEOREM XVI. *In any equilibrium configuration of prescribed mass and radius in which both* ρ *and* $K=\frac{P}{\rho^{1+\frac{1}{m}}}$ $(0< m \leq 3)$ *do not increase outward, the minimum value of* K_c *is the constant value of* K *which must be ascribed to a complete polytrope of index* m *having the given mass and radius.*

Unlike Chandrasekhar's, our proof makes no use of computations, and is a straight-forward deduction from the definition of a variable polytrope and the very general theorem of Candler.[2]

We can similarly prove the following theorem due to Chandrasekhar[4] :

In a gaseous stellar configuration in which both ρ *and* $1-\beta$ *do not increase outward,* $(1-\beta_c)$ *must be greater than the constant value of* $1-\beta$ *ascribed to a standard model configuration of the same mass.*

This will be clear when we consider that the necessary and sufficient condition for $(1-\beta)$ not to increase outward is that the polytropic index $n \leq 3$, as proved in Theorem X. We have further shown in Theorem VIII, Cor. 1, that β_c is maximum for maximum n. Considering polytropes of index $n \leq 3$, the above theorem of Chandrasekhar follows.

The theorem just quoted does not hold for polytropes with $n>3$. For such polytropes we can, by a method similar to that adopted above, prove the following analogous theorem.

THEOREM XVII. *For gaseous spherical configurations in which both* ρ *and* β *do not increase outward, the minimum value of* β_c *is attained in the standard model configuration of the same mass.*

We have already proved that β^* of equation (38) is the β_c for a uniform density star, and, in Theorem VIII, Cor. 1, that β_c decreases with n. The Theorem XVII, therefore, gives, for polytropes with $n > 3$, a closer upper limit to $(1-\beta_c)$ than the one given in the Theorem 7 of Chandrasekhar,[3] and quoted in equations (37) and (38) of our paper.

It thus appears that the standard model $(n=3)$ is the limiting configuration both for polytropes with $n<3$ as well as for those with $n>3$. This accords well with the critical nature of the standard model.

It may be mentioned here that Chandrasekhar[4] derives equations comparable to the standard formulæ of Eddington's theory[6] and remarks,

"The equations in that theory now become inequalities. This makes the conclusions drawn on the basis of the standard model have a 'minimal' character which is of considerable physical importance."

Candler's theorems[2] enable us to enunciate and prove extremal properties of K as defined by (50) without Chandrasekhar's restriction[4] of K decreasing outward.

THEOREM XVIII. *If the polytropic indices n, n^1 of two polytropes S, S^1 with the same central pressure and density vary in any manner between the centre and surface such that $-1<n<n^1\leq 5$ at all corresponding points, then* $k=\dfrac{P}{\rho^{1+\frac{1}{m}}}$ $(0<m\leq 3)$ *is less at any point in S (except the centre) than at the corresponding point in S^1.*

From Candler's[2] equations (42) and (43), we have

$$\frac{P/\rho^{\nu}}{P^1/\rho^{1\nu}}=\left(\frac{Y^1}{Y}\right)^{\nu}\exp\int_0^X\frac{\{(4-3\nu)X^2-3\nu\}(Y-Y^1)dX}{\{(3+x^2)Y-2X\}\{(3+X^2)Y^1-2X\}}, \quad (54)$$

where $\nu = 1 + \dfrac{1}{m}$.

The ratio (54) <1, for $\nu\geq\frac{4}{3}$ and for all $X\neq 0$.

COR. *Under the conditions of the Theorem,* $K=\dfrac{P}{\rho^{1+\frac{1}{m}}}$ $(o<m\leq 3)$ *for a variable polytrope lies between its values at the corresponding points of the limiting, uniform polytropes of the same central pressure and density.*

We can similarly prove the following theorem for polytropes of fixed mass and radius.

THEOREM XIX. *If the polytropic indices n, n^1 of two polytropes S, S^1 with the same mass and radius vary in any manner between the centre and surface such that $-1<n<n^1\leq 5$ at all corresponding points, then* $K=\dfrac{P}{\rho^{1+\frac{1}{m}}}$ $(o<m\leq 3)$ *is greater at any point in S than at the corresponding point in S^1 when X is small, and the converse when X is large.*

COR. 1. *Under the conditions of the Theorem,* $K=\dfrac{P}{\rho^{1+\frac{1}{m}}}$ $(0<m\leq 3)$ *for a variable polytrope, for small (or large) X, lies between its values at the corresponding points of the limiting, uniform polytropes of the same mass and radius.*

COR. 2. *If S be a polytrope whose index varies between the limits 0 and 5, then*

$$K_c=\frac{P_c}{\rho_c^{1+\frac{1}{m}}}\leq\tfrac{1}{2}\left(\frac{4\pi}{3}\right)^{\frac{1}{m}}G\,M^{\frac{m-1}{m}}R^{\frac{3-m}{m}}, (o<m\leq 3) \quad . \quad . \quad (55)$$

It follows from Theorem XIX that the maximum value of K_c is attained in the sphere of uniform density ($n=0$), and, by (39), the right-hand side of (55) gives the value of K_c for a uniform density sphere.

In equations (129) and (130) of his monograph,[3] Chandrasekhar has given the following upper limit for K_c :

$$K_c \leq \left(\frac{4\pi}{3}\right)^{\frac{1}{m}} \frac{m}{3(m-1)} \, GR^{\frac{3-m}{m}} M^{\frac{m-1}{m}}, \, (1 < m \leq 3). \qquad (56)$$

The inequality (55) is more general than (56) and gives a closer (in fact the best possible) upper limit.

The upper limits given in (55) and (56) coincide when $m=3$, and in this case it is easily seen that they give Chandrasekhar's Theorem 7 on the upper limit of the central radiation pressure for a star of given mass,[3] as expressed in equations (37) and (38) of our paper.

The above investigation has been carried out under the guidance of Prof. A. C. Banerji, to whom the author's most respectful thanks are due.

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SOLUTIONS OF THE DIFFERENTIAL EQUATIONS $f^r(x)=f\left(\pm\frac{1}{x}\right)$ WHERE $f\left(\pm\frac{1}{x}\right)$ ARE PROPERLY DEFINED

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(Dr. Ludwik Silberstein solved in a recent paper[1] the differential equation $f^1(x)= f\left(\frac{1}{x}\right)$ and this tempted me to tackle the above problem, which is nothing but a generalisation of his case.)

I. First take the case $f^r(x) = f\left(\frac{1}{x}\right)$.

The equation reads: the rth derivative of a function at x is equal to the value of the function at $\frac{1}{x}$. Thus unless $x=1$, the proposed equation is not an ordinary differential equation.

To satisfy it put $f(x) = x^m+\lambda x^n$
where m, n & λ are constants

whereby we get $m(m-1)(m-2)\ldots\ldots(m-r+1)x^{m-r}+\lambda n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-r+1)x^{n-r} = x^{-m}+\lambda x^{-n}$, which becomes an identity.

Whence

$m(m-1)(m-2)\ldots\ldots(m-r+1) = \lambda\ldots\ldots(i)$ $\qquad m-r= -n\ldots\ldots(ii)$

$\lambda n(n-1)(n-2)\ldots\ldots(n-r+1) = 1\ldots\ldots(iii)$ $\qquad n-r= -m\ldots\ldots(iv)$

Now (ii) & (iv) give the same result, *viz.*, $n=r-m\ldots\ldots(v)$.

Combining (i) & (iii) we find

$$m(m-1)(m-2)\ldots\ldots(m-r+1).\ n(n-1)(n-2)\ldots\ldots(n-r+1)=1.$$

Substituting for n from (v) we get

$$m(m-1)^2(m-2)^2\ldots\ldots(m-r+1)^2(m-r) = (-1)^r\ldots\ldots(vi)$$

Now this gives $2r$ values of m and thereby $2r$ values of n from (v).

Now λ is given by (i).

So all the constants are now determined.

Therefore the complete solution of the differential equation is given by

$$f(x)=\sum_{p=1}^{2r} A_p \left\{ x^{m_p}+m_p\,(m_p-1)\ldots\ldots(m_p-r+1)x^{r-m_p} \right\} \ldots\ldots(\Omega)$$

where A's are arbitrary constants.

A point may arise as to why I do not assume $f(x)=x^m+\lambda x^n+\mu x^l+\ldots\ldots\ldots\ldots$

But it may be proved easily that in that case we get one more equation than the number of unknowns and no two of those equations coincide as it has been found in the case of (*ii*) & (*iv*). Hence etc.

Particular Case:—

(A) *When* $r=1$, the differential equation becomes

$$f^1(x)=f\left(\frac{1}{x}\right)$$

so that our equations for determining m, n & λ become

$$\begin{matrix} m=\lambda. \\ n\lambda=1, \end{matrix} \ \& \ -n=m-1$$

or $\quad m(m-1)=-1 \quad$ or $\quad m^2-m+1=0.$

$$\therefore \quad m=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm i\sqrt{3}}{2},\ i.e.,\ m_1=\frac{1+i\sqrt{3}}{2}=\frac{1}{m_2}.$$

and our solution becomes

$$f(x)=A_1\left(x^{m_1}+m_1x^{\frac{1}{m_1}}\right)+A_2\left(x^{m_2}+m_2x^{\frac{1}{m_2}}\right)$$

where A_1 & A_2 are arbitrary constants.

$$=A_1\left(x^{m_1}+m_1x^{\frac{1}{m_1}}\right)+A_2\left(x^{\frac{1}{m_1}}+\frac{1}{m_1}x^{m_1}\right)$$

$$=\left(A_1+\frac{A_2}{m_1}\right)\left(x^{m_1}+m_1x^{\frac{1}{m_1}}\right)$$

$$=a\left(x^{m_1}+m_1x^{\frac{1}{m_1}}\right),$$

where a is another arbitrary constant and here stands for $\left(A_1+\frac{A_2}{m_1}\right)$.

This is the result arrived at by Silberstein.

After this he has simplified it further and his work runs as follows :—

[Put $x = e^{t}$.

$$\frac{1}{a} f(t) = e^{mt} + m e^{\frac{t}{m}} = e^{\frac{t}{2}} \left\{ e^{\frac{\sqrt{3}}{2}it} + m e^{-\frac{\sqrt{3}}{2}it} \right\}$$

or after simple reductions,

$$f(t) = a e^{\frac{t}{2}} \left(\sqrt{3} \cos \frac{\sqrt{3}}{2} t + \sin \frac{\sqrt{3}}{2} t \right) \quad . \quad . \quad . \quad (3)$$

In terms of x itself,

$$(x) = a\sqrt{x} \left\{ \sqrt{3} \cos\left(\frac{\sqrt{3}}{2} \log x \right) + \sin \left(\frac{\sqrt{3}}{2} \log x \right) \right\}$$

Since $\sqrt{3} \cos \frac{\sqrt{3}}{2} t + \sin \frac{\sqrt{3}}{2} t = 2 \cos\left(\frac{\sqrt{3}}{2} t - \frac{\pi}{3} \right)$

We may write also $f(x) = a\sqrt{x} \cos \left(\frac{\sqrt{3}}{2} \log x - \frac{\pi}{3} \right)$. . . . . . (4)

This, with an arbitrary constant coefficient a, is the complete solution of the equation $f^1(x) = f\left(\frac{1}{x}\right)$.

For $x=1$, $f(1) = a \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2} a$

so that $\quad f(x) = \frac{2}{\sqrt{3}} f(1) \sqrt{x} \cos \left(\frac{\sqrt{3}}{2} \log x - \frac{\pi}{3} \right)$

The function is completely determined by its (arbitrarily given) value at $x=1$.]

Now there are some minor points or slips, in Silberstein's treatment, which it is my duty to point out.

When he puts $x = e^{t}$, $f(x)$ cannot be equal to $f(t)$, but $f(x)$ must change to a function of t, say, $F(t)$, where $F(t) = f(e^{t})$.

Again, he is not justified in writing a for the arbitrary constant throughout, for example in (3) of his treatment, his "a" actually is equal to a_1,

where $a \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) = a_1$, this a occurs in (1) or explicitly the a

in $f(x) = a \left(x^{m} + m x^{\frac{1}{m}} \right)$

Again his 'a' in (4) is not the same as his 'a' either in (3) or (1).

But it is, in fact, a_2, another arbitrary constant, where $a_2 = 2a_1$.

If we retain the same a throughout, it is very likely to create confusion.

There is another point, namely, all his $\frac{\pi}{3}$'s must be changed to '$\frac{\pi}{6}$'s for $\sqrt{3}\cos\frac{\sqrt{3}}{2}t+\sin\frac{\sqrt{3}}{2}t=2\cos\left(\frac{\sqrt{3}}{2}t-\frac{\pi}{6}\right)$ and not $2\cos\left(\frac{\sqrt{3}}{2}t-\frac{\pi}{3}\right)$ as found in the paper.

II. Now consider the Equation $f^r(x)=f\left(-\frac{1}{x}\right)$.

Proceeding as in Case I, and putting $f(x)=x^m+\lambda x^n$ we get the equations for determining m, n and λ to be

$$n=r-m \qquad \text{(i)}$$

$$m(m-1)(m-2)\ldots\ldots(m-r+1)=\lambda(-1)^n \qquad \text{(ii)}$$

and

$$2\lambda n(n-1)(n-2)\ldots\ldots(n-r+1)=(-1)^m \qquad \text{(iii)}$$

Combining (*ii*) & (*iii*) we get

$$m(m-1)(m-2)\ldots\ldots(m-r+1)\,n(n-1)(n-2)\ldots(n-r+1)=(-1)^{m+n}$$
$$=(-1)^r\ldots\text{by (i)}$$

$$\text{or}\quad m(m-1)^2(m-2)^2\ldots\ldots(m-r+1)^2(m-r)=1 \text{ by (i)}$$

which gives $2r$ values of m

$\therefore$ the complete solution of the above equation is given by

$$f(x)=\sum_{p=1}^{2r}B_p\left[x^{m_p}+(-1)^{r-m_p}m_p(m_p-1)(m_p-2)\ldots(m_p-r+1)x^{r-m_p}\right]$$

where B's are arbitrary constants.

Particular Cases of I :—

(B) *When $r=2$*, the equation for determining m is given by

$m(m-1)^2(m-2)=(-1)^2$

or $(m^2-2m)(m^2-2m+1)=1$

or $y(y+1)=1$ where $y=m^2-2m$

or $y^2+y-1=0$ which gives $y=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$

whence we get $m^2-2m-\left(\frac{-1+\sqrt{5}}{2}\right)=0$ or $2m^2-4m-(\sqrt{5}-1)=0$

or $m=\frac{4\pm2\sqrt{4+2(\sqrt{5}-1)}}{4}=\frac{2\pm\sqrt{2\sqrt{5}+2}}{2}$

and $m^2-2m-\left(\frac{-1-\sqrt{5}}{2}\right)=0$

or $2m^2-4m+(\sqrt{5}+1)=0$

or $m=\frac{4\pm2\sqrt{4-2(\sqrt{5}+1)}}{2}=\frac{2\pm\sqrt{2-2\sqrt{5}}}{2}$

Thus we get the *four* values of m and thus the complete solution by substituting in (Ω).

(C) *When $r=3$,*

$$m(m-1)^2\,(m-2)^2\,(m-3)=(-1)^3$$

$$\text{or } (m^2-3m)\,(m^2-3m+2)^2=-1$$

$$\text{or } y(y+2)^2+1=0. \qquad \text{where } y=m^2-3m$$

$$\text{or } y^3+4y^2+4y+1=0$$

$$\text{or } (y+1)\,(y^2-y+1)+4y\,(y+1)=0$$

$$\text{or } (y+1)\,(y^2+3y+1)=0 \text{ which gives } y=-1 \text{ or } \frac{-3\pm\sqrt{9-4}}{2}, \textit{ i.e., } \frac{-3\pm\sqrt{5}}{2}$$

When $y=-1$ or $m^2-3m+1=0$ or $m=\dfrac{3\pm\sqrt{9-4}}{2}=\dfrac{3\pm\sqrt{5}}{2}$,

$$y=\frac{-3+\sqrt{5}}{2} \text{ or } 2m^2-6m-(-3+\sqrt{5})=0$$

$$\text{or } m=\frac{6\pm\sqrt{36+8(-3+\sqrt{5})}}{4}=\frac{3\pm\sqrt{9+2\sqrt{5}-6}}{2}=\frac{3\pm\sqrt{3+2\sqrt{5}}}{2}$$

$$y=\frac{-3-\sqrt{5}}{2} \text{ or } 2m^2-6m+(3+\sqrt{5})=0$$

$$\text{or } m=\frac{6\pm\sqrt{36-8(3+\sqrt{5})}}{4}=\frac{3\pm\sqrt{3-2\sqrt{5}}}{2}$$

This shows that we get the *six* concrete values of m and after substituting these values in (Ω) we shall get the complete solution.

(D) *When $r=4$*

$$m(m-1)^2\,(m-2)^2\,(m-3)^2\,(m-4)=(-1)^4$$

$$\text{or } (m^2-4m)\,(m^2-4m+3)^2\,(m^2-4m+4)=1$$

$$\text{or } y\,(y+4)\,(y+3)^2-1=0 \quad \text{where } y=m^2-4m$$

$$\text{or } (y^2+4y)\,(y^2+6y+9)-1=0$$

$$\text{or } y^4+10y^3+33y^2+36y-1=0 \qquad \ldots\ldots\ (i).$$

Suppose this equation to be the same as

$$ay^4+4by^3+6cy^2+4dy+e=0$$

Comparing these two we get $a=1$, $b=\frac{5}{2}$, $c=\frac{11}{2}$, $d=9$, and $e=-1$.

Putting $z=y+\frac{5}{2}$ in (i) we get

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0,$$

where $H\equiv ac-b^2=\frac{11}{2}-\frac{25}{4}=-\frac{3}{4}$

$I\equiv ae-4bd+3c^2=-1-90+\frac{363}{4}=-\frac{1}{4}$

F. 6

$$G \equiv a^2d - 3abc + 2b^3 = 9 - \tfrac{165}{4} + \tfrac{125}{4} = -1$$

$$a^2I - 3H^2 = -\tfrac{1}{4} - 3\left(\tfrac{9}{16}\right) = -\tfrac{1}{4} - \tfrac{27}{16} = -\tfrac{31}{16}$$

$$J \equiv ace + 2bcd - ad^2 - eb^2 - c^3$$

$$= -\tfrac{11}{2} + \tfrac{495}{2} - 81 + \tfrac{25}{4} - \tfrac{1331}{8} = \frac{-44 + 1980 - 648 + 50 - 1331}{8} = \tfrac{7}{8}.$$

$$-G^2 \equiv 4H^3 - a^2HI + a^3J.$$

We find[3] that if $z = \sqrt{p} + \sqrt{q} + \sqrt{r}$ satisfy the above equation. p, q and r must be the roots of

$$t^3 - 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0 \quad . \quad . \quad . \qquad (ii)$$

Or since $-G^2 \equiv 4H^3 - a^2HI + a^3J$, where $J \equiv ace + 2bcd - ad^2 - eb^2 - c^3$, the equation (ii) may be written in the form

$$4(t+H)^3 - a^2I(t+H) + a^3J = 0$$

Now putting $t + H = \theta a^2$ . . . . (iii)

we get

$$4\theta^3 a^3 - Ia\theta + J = 0$$

or

$$4\theta^3 + \frac{\theta}{4} + \frac{7}{8} = 0$$

or

$$\theta^3 + 3H_1\theta + G_1 = 0 \quad . \quad . \quad . \quad (iv), \text{ where } H_1 = \tfrac{1}{48},\ G_1 = \tfrac{7}{32}.$$

Since $G_1{}^2 + 4H_1{}^3$ is positive two roots of θ will evidently be imaginary[4].

Now[5] if $\theta = \sqrt[3]{p_1} + \sqrt[3]{q_1}$ is a root of (iv), we must have

$$p_1 = \tfrac{1}{2}\left(-G_1 + \sqrt{G_1{}^2 + 4H_1{}^3}\right), \text{ and } q_1 = \tfrac{1}{2}\left(-G_1 - \sqrt{G_1{}^2 + 4H_1{}^3}\right).$$

so that if θ_1, θ_2 and θ_3 be the three roots of (iv), we have

$$\theta_1 = \sqrt[3]{p_1} + \frac{-H_1}{\sqrt[3]{p_1}},\ \theta_2 = \omega\sqrt[3]{p_1} + \omega^2\frac{-H1}{\sqrt[3]{p_1}}, \text{ and } \theta_3 = \omega^2\sqrt[3]{p_1} + \omega\frac{-H_1}{\sqrt[3]{p_1}},$$

where ω, ω^2 are cube roots of unity.

From (iii) we get $t = \theta - H$, *i.e.*, $p = \theta_1 - H$, $q = \theta_2 - H$, $r = \theta_3 - H$

so that we get $\quad m^2 - 4m = y = z_s - \tfrac{5}{2}$, $s = 1, 2, 3, 4$

where

$$z_1 = \sqrt{p} + \sqrt{q} + \sqrt{r}$$
$$z_2 = +\sqrt{p} - \sqrt{q} - \sqrt{r}$$
$$z_3 = -\sqrt{p} + \sqrt{q} - \sqrt{r}$$
$$z_4 = -\sqrt{p} - \sqrt{q} + \sqrt{r}$$

Thus we shall get *eight* roots of m and after proper substitution we shall get the complete solution.

(E) *When* $r=5$,

$$m(m-1)^2(m-2)^2(m-3)^2(m-4)^2(m-5)=(-1)^5$$

or $$(m^2-5m)\,(m^2-5m+4)^2(m^2-5m+6)^2+1=0$$

or $$y(y+4)^2(y+6)^2+1=0 \qquad \text{where } y=m^2-5m$$

or $$(z-5)\,(z^2-1)^2+1=0 \qquad \text{where } y+5=z$$

or $$(z-5)\,(z^4-2z^2+1)+1=0$$

or $$z^5-5z^4-2z^3+10z^2+z-4=0.$$

Putting $z-1=t$, we get

$$f(t)\equiv t^5-12t^3-16t^2+1=0$$

Also consider $f(-t)=0$ or $\phi(t)=0$

whereby $$\phi(t)\equiv t^5-12t^3+16t^2-1=0$$

By Descarte's Rule of Signs[6], we may say that $f(t)=0$ can have, at most, *two* positive and *three* negative roots.

Now $$\left.\begin{aligned} f(0)&=1 \\ f(1)&=-26 \end{aligned}\right\} \left.\begin{aligned} f(3)&=243-324-144+1=\text{ a negative quantity} \\ f(4)&=1024-768-256+1=1 \end{aligned}\right\}$$

$$\left.\begin{aligned} \phi(0)&=-1 \\ \phi(1)&=4 \end{aligned}\right\} \left.\begin{aligned} \phi(2)&=-1 \\ \phi(3)&=62 \end{aligned}\right\}$$

These indicate that $f(t)$ has two real positive roots, one lying between 0 and 1, and the other between 3 and 4 and three negative roots, one lying between 0 and -1, another between -1 and -2, and finally the last one between -2 and -3 and thus all the roots of $f(t)$ are real.

Approximations to these roots can very easily be found by applying "Newton's Method of Approximation."[7]

(F) *When* $r=6$,

$$m(m-1)^2(m-2)^2(m-3)^2(m-4)^2(m-5)^2(m-6)=(-1)^6$$

or $$(m^2-6m)\,(m^2-6m+5)^2(m^2-6m+8)^2(m^2-6m+9)=1$$

or $$y(y+9)\,(y+5)^2(y+8)^2-1=0 \qquad \text{where } y=m^2-6m$$

or $$z^2(z-5)\,(z+4)\,(z+3)^2-1=0 \qquad \text{where } z=y+5$$

so that $$f(z)\equiv\{z(z+3)\}^2\,(z+4)\,(z-5)-1=0$$

It is evident that $f(z)$ has a positive root lying between 5 and 6 and a negative root between -4 and -5.

Expanding $f(z)$ we get $$f(z)\equiv z^6+5z^5-17z^4-129z^3-180z^2-1=0$$
$$f(-z)\equiv z^6-5z^5-17z^4+129z^3-180z^2-1=0.$$

Using Descarte's Rule of Signs, we can say there are, *at most*, one positive root and three negative real roots, and thereby there must be *at least* two imaginary roots

Arguing further, from $f(z)=0$, we can infer that there is *one* positive root, *one* negative root and four imaginary ones. Then putting each $=y=m^2-6m$, we shall get *12* roots of m, whereby the complete solution can be found, though our solution will only be approximate.

(G) Now taking the general case,

$$m(m-1)^2(m-2)^2\ldots\ldots(m-r+1)^2(m-r)=(-1)^r \qquad . \quad . \quad (\text{M})$$

This is an equation of $(2r)^{th}$ degree in m. Putting $y=m^2-mr$, this can be reduced to an equation of the r^{th} degree and the method of combining the factors on the Left Hand Side of equation (M) is that we group together the first and the last, the second and the last but one, the third and the last but two and so on.

In every case when r is even, m must have two real roots, one between *0* and -1 and the other between r and $r+1$ and the other roots will be imaginary.

Since there is no general method of solving a general equation of the fifth degree and higher orders, we have no other way but to take to approximations in the case $r=5$ and onwards.

Again in cases when r is odd, all the roots of y will be real and the approximations of roots may be found in the following way :—

When r is odd, suppose it to be equal to $2s+1$.

Now

$$m(m-1)^2(m-2)^2.(m-3)^2(m-4)^2\ldots\ldots(m-r+3)^2(m-r+2)^2(m-r+1)^2(m-r)=(-1)^r.$$

or $\quad y(y+r-1)^2\{y+2(r-2)\}^2\{y+3(r-3)\}^2\{y+4(r-4)\}^2\ldots\ldots\{y+s(s+1)^2=-1$

or $\quad y(y+r-1)^2\{y+2(r-2)\}^2\{y+3(r-3)\}^2\ldots\ldots\left\{y+\frac{r-1}{2}.\frac{r+1}{2}\right\}^2=-1.$

The approximate values of y are given by

$$y=-\frac{1}{(r-1)^2\cdot\{2(r-2)\}^2\ldots\left(\frac{r-1}{2}\cdot\frac{r+1}{2}\right)^2}.$$

$$y+r-1=\pm\frac{1}{\sqrt{r-1}\,\{(1-r)+2(r-2)\}\{(1-r)+3(r-3)\}\ldots\ldots\left\{(1-r)+\frac{r-1}{2}.\frac{r+1}{2}\right\}}$$

$$y+2(r-2)=\pm\frac{1}{\sqrt{2(r-2)}\{2(2-r)+r-1\}\{2(2-r)+3(r-3)\}\ldots\left\{2(2-r)+\frac{r-1}{2}\cdot\frac{r+1}{2}\right\}}$$

. . . .

. . . .

$$y+\frac{r-1}{2}\cdot\frac{r+1}{2}=\pm\frac{1}{\sqrt{\frac{r-1}{2}\cdot\frac{r+1}{2}}}\cdot\frac{1}{\left\{-\frac{r-1}{2}\cdot\frac{r+1}{2}+r-1\right\}\left\{-\frac{r-1}{2}\cdot\frac{r+1}{2}+2(r-2)\right\}}$$

$$\ldots\ldots \text{to } \left(\frac{r-1}{2}\right) \text{ factors}$$

The equation II can be treated in a similar way.

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SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS ARISING IN PROBLEMS OF VARYING VISCOSITY IN HYDRODYNAMICS

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INTRODUCTION

These are some of the Differential Equations which I came across while tackling problems of varying viscosity in Hydrodynamics. Want of definite solutions of these equations led me to work on this paper.

I.

$$\frac{\partial \xi}{\partial t}=(\nu_0+\beta_1 x)\ \nabla^2\xi, \text{ where } \nu_0 \text{ and } \beta_1 \text{ are finite constants.}$$

Put $\xi=e^{\lambda t}\,\xi_1$ . . . . (1)

where ξ_1 is a function of x, y and z but independent of t and λ is a finite constant, so that we get

$$\lambda e^{\lambda t}\,\xi_1=(\nu_0+\beta_1 x)\ e^{\lambda t}\ \nabla^2\xi_1$$

$$\text{or } \nabla^2\xi_1-\frac{\lambda}{\nu_0+\beta_1 x}\xi_1=0$$

$$\text{or } \frac{\partial^2\xi_1}{\partial x^2}+\frac{\partial^2\xi_1}{\partial y^2}+\frac{\partial^2\xi_1}{\partial z^2}-\frac{\lambda}{\nu_0+\beta_1 x}\xi_1=0 \quad \ldots \quad (2)$$

Putting $\nu_0+\beta_1 x=\beta_1 X$ . . . . (3)

or $\partial x=\partial X$ and changing y to Y and z to Z in (2) we get

$$\frac{\partial^2\xi_1}{\partial Y^3}+\frac{\partial^2\xi_1}{\partial Z^2}+\frac{\partial^2\xi_1}{\partial X^2}-\frac{\lambda}{\beta_1 X}\xi_1=0$$

$$\text{or } \frac{\partial^2\xi_1}{\partial Y^2}+\frac{\partial^2\xi_1}{\partial Z^2}+\frac{\partial^2\xi_1}{\partial X^2}-\frac{\lambda_1}{X}\ \xi_1=0 \quad \ldots \quad (4)$$

where $$\frac{\lambda}{\beta_1}=\lambda_1 \quad \ldots \quad (5)$$

Now put $\xi_1=\xi_2\,R$ . . . . (6)

where ξ_2 is independent of X and a function of Y and Z whereas R is a function of X alone. This substitution leads to

$$R\left(\frac{\partial^2\xi_2}{\partial Y^2}+\frac{\partial^2\xi_2}{\partial Z^2}\right)+\xi_2\left(\frac{\partial^2 R}{\partial X^2}-\frac{\lambda_1}{X}R\right)=0$$

$$\text{or } \frac{1}{\xi_2}\left(\frac{\partial^2\xi_2}{\partial Y^2}+\frac{\partial^2\xi_2}{\partial Z^2}\right)+\left(\frac{1}{R}\frac{\partial^2 R}{\partial X^2}-\frac{\lambda_1}{X}\right)=0 \qquad (7)$$

Equation (7) is satisfied if we write

$$\left.\begin{array}{lr} \dfrac{\partial^2\xi_2}{\partial Y^2}+\dfrac{\partial^2\xi_2}{\partial Z^2}+k\xi_2=0 & (8) \\ \text{and} \quad \dfrac{1}{R}\dfrac{d^2R}{dX^2}-\dfrac{\lambda_1}{X}=k & (9) \end{array}\right\} \text{where } k \text{ is a finite constant.}$$

Let us first consider equation (9).

$$\frac{1}{R}\frac{d^2R}{dX^2}-\frac{\lambda_1}{X}=k$$

$$\text{or } X\frac{d^2R}{dX^2}-(\lambda_1+kX)R=0.$$

Suppose $R=\sum_{p=0}^{\infty} A_p \; X^{m+p}$

Substituting this value of R in the above equation, we get

$$X\left[A_0 m(m-1)X^{m-2}+A_1(m+1)mX^{m-1}+A_2(m+2)(m+1)X^m + \ldots + A_p(m+p)(m+p-1)X^{m+p-2}+\ldots\right]$$
$$-(\lambda_1+kX)\left(A_0X^m+A_1X^{m+1}+A_2X^{m+2}+\ldots\ldots+A_pX^{m+p}+\ldots\ldots\right)=0$$

$$\text{or}\left.\begin{array}{r} A_0m(m-1)X^{m-1}+A_1(m+1)mX^m+A_2(m+2)(m+1)X^{m+1}+\ldots \\ +A_{p+1}(m+p+1)(m+p)X^{m+p}+\ldots \\ -A_0\lambda_1X^m-A_1\lambda_1X^{m+1}\ldots\ldots\ldots-A_p\lambda_1X^{m+p}-\ldots\ldots \\ -A_0kX^{m+1}\ldots\ldots\ldots-A_{p-1}k.X^{m+p}-\ldots \end{array}\right\}=0$$

whence

$$m(m-1)=0 \qquad (10)$$

$$A_1m(m+1)-A_0\lambda_1=0 \qquad (11)$$

$$A_{p+1}(m+p+1)(m+p)-A_p\lambda_1-A_{p-1}k=0 \qquad (12)$$

From (12) we get

$$\frac{A_{p+1}}{A_{p-1}}=\frac{k}{(m+p+1)(m+p)}+\frac{\lambda_1}{(m+p+1)(m+p)}\cdot\frac{A_p}{A_{p-1}},\ \text{for } p\geq 1$$

Let $$\underset{p\to\infty}{\text{Lt}}\frac{A_p}{A_{p-1}}=k_0$$

where $$k_0\leq 1.$$

Then $$\frac{A_p}{A_{p-1}}=k_0+\varepsilon \qquad \text{where } \varepsilon\to 0$$

and $$\frac{A_{p+1}}{A_{p-1}}=\frac{A_{p+1}}{A_p}\cdot\frac{A_p}{A_{p-1}}=(k_0+\varepsilon_1)(k_0+\varepsilon) \quad \text{where } \varepsilon_1\to 0$$

Therefore proceeding to limits as $p\to\infty$
we find $k_0=0$

$$\therefore \quad \frac{A_{p+1}}{A_p}\longrightarrow 0 \text{ as } p\longrightarrow\infty.$$

From this result we may conclude that the series thus found for $m=0$ or 1 will be absolutely and uniformly convergent for all values of x including those in the neighbourhood of $x=0$, which is a regular[1] singular point for the differential equation.

Since the difference of the two values of m is 1, which is an integer, the two solutions, *viz.*, R_1 and R_2 will be given by $R_1=\Big[R\Big]_{m=1}$ and $R_2=\Big[\frac{\partial R}{\partial m}\Big]_{m=0}$, by the method of Frobenius.

Let us first find R_1.

Now $A_1\, m(m+1)-A_0\lambda_1=0.$ . . . . (11)

$A_{p+1}\ (m+p+1)(m+p)-A_p\,\lambda_1-A_{p-1}\,k=0$ . . . . . . (12)

for $p\geq 1$.

$A_1=\frac{A_0\,\lambda_1}{m\,(m+1)}=\frac{B_0\,\lambda_1}{m+1}$ where B_0 and A_0 are arbitrary constants, related by $B_0 m=A_0$.

Putting $p=1$ in (12), $A_2\ (m+2)(m+1)-A_1\,\lambda_1-A_0 k=0$

or $$A_2\,(m+2)(m+1)=B_0\left(\frac{\lambda_1^{\,2}}{m+1}+mk\right)$$

or $$A_2=\frac{B_0}{(m+2)(m+1)}\left(\frac{\lambda_1^{\,2}}{m+1}+mk\right)$$

$$A_3=\frac{1}{(m+3)(m+2)}(A_2\,\lambda_1+A_1\,k)=\frac{B_0\,\lambda_1}{(m+3)(m+2)}\left\{\frac{1}{(m+2)(m+1)}\left(\frac{\lambda_1^{\,2}}{m+1}+mk\right)+\frac{k}{m+1}\right\}$$

$$=\frac{B_0\,\lambda_1}{(m+3)(m+2)(m+1)}\cdot\left\{\frac{1}{m+2}\left(\frac{\lambda_1^{\,2}}{m+1}+mk\right)+k\right\}$$

$$A_4 = \frac{1}{(m+4)(m+3)}(A_3\lambda_1 + A_2 k) = \frac{1}{(m+4)(m+3)}\left[\frac{B_0\lambda_1^2}{(m+3)(m+2)(m+1)}\right.$$
$$\left.\left\{\frac{1}{m+2}\left(\frac{\lambda_1^2}{m+1}+mk\right)+k\right\} + \frac{B_0 k}{(m+2)(m+1)}\left(\frac{\lambda_1^2}{m+1}+mk\right)\right]$$
$$= \frac{B_0}{(m+4)(m+3)(m+2)(m+1)}\left[\frac{\lambda_1^2}{m+3}\cdot\left\{\frac{1}{m+2}\left(\frac{\lambda_1^2}{m+1}+mk\right)+k\right\}\right.$$
$$\left.+ k\left(\frac{\lambda_1^2}{m+1}+mk\right)\right], \text{ etc.}$$

Thus

$$R = B_0 X^m\left[m + \frac{\lambda_1}{m+1}X + \frac{1}{(m+2)(m+1)}\left(\frac{\lambda_1^2}{m+1}+mk\right)X^2\right.$$
$$+ \frac{\lambda_1}{(m+3)(m+2)(m+1)}\left\{\frac{1}{m+2}\left(\frac{\lambda_1^2}{m+1}+mk\right)+k\right\}X^3$$
$$+ \frac{1}{(m+4)(m+3)(m+2)(m+1)}\left\{\frac{\lambda_1^2}{m+3}\cdot\overline{\frac{1}{m+2}\left(\frac{\lambda_1^2}{m+1}+mk\right)+k}\right.$$
$$\left.+ k\left(\frac{\lambda_1^2}{m+1}+mk\right)X^4 + \ldots\ldots\right]$$

so that $R_1 = [R]_{m=1}$

$$R_2 = \left[\frac{\partial R}{\partial m}\right]_{m=0}$$

Hence, the complete solution is $R=(R_1+R_2)$

The form of R_1 is given by

$$R_1 = B_0 X\left[1 + \frac{\lambda_1}{1\cdot 2}\cdot X + \left(\frac{\lambda_1}{2\cdot 3}\cdot\frac{\lambda_1}{1\cdot 2}+\frac{k}{2\cdot 3}\right)X^2\right.$$
$$+ \frac{\lambda_1}{3\cdot 4}\left\{\left(\frac{\lambda_1}{2\cdot 3}\cdot\frac{\lambda_1}{1\cdot 2}+\frac{k}{2\cdot 3}\right)+\frac{k\cdot\lambda_1}{1\cdot 2}\right\}X^3$$
$$+ \frac{\lambda_1}{4\cdot 5}\left[\left\{\frac{\lambda_1}{3\cdot 4}\left(\frac{\lambda_1}{2\cdot 3}\cdot\frac{\lambda_1}{1\cdot 2}+\frac{k}{2\cdot 3}\right)+k\cdot\frac{\lambda_1}{1\cdot 2}\right\}\right.$$
$$\left.\left.+ k\left(\frac{\lambda_1}{2\cdot 3}\cdot\frac{\lambda_1}{1\cdot 2}+\frac{k}{2\cdot 3}\right)\right]X^4 + \ldots\ldots\right]$$

Now take equation (8).

$$\frac{\partial^2\xi_2}{\partial y^2} + \frac{\partial^2\xi_2}{\partial z^2} + k\xi_2 = 0 \qquad \text{Suppose } \begin{matrix} Y=y=r\sin\theta \\ Z=z=r\cos\theta \end{matrix}\Bigg\}$$

Transforming to polars we get

$$\frac{\partial^2\xi_2}{\partial r^2} + \frac{1}{r}\frac{\partial\xi_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2\xi_2}{\partial\theta^2} + k\xi_2 = 0$$

Put $\xi_2 = \cos\mu\theta\eta$. . . . . (15)

where η is a function of r alone, and μ is a finite constant, so we have

$$\cos\mu\theta\left[\frac{\partial^2\eta}{\partial r^2}+\frac{1}{r}\frac{\partial\eta}{\partial r}-\frac{\mu^2}{r^2}\eta+k\eta\right]=0$$

$$\text{or } \frac{d^2\eta}{dr^2}+\frac{1}{r}\frac{d\eta}{dr}+\left(-\frac{\mu^2}{r^2}+k\right)\eta=0$$

$$\text{or } r^2\frac{d^2\eta}{dr^2}+r\frac{d\eta}{dr}+(kr^2-\mu^2)\eta=0 \qquad . \quad . \quad . \quad . \quad . \quad (16)$$

Here two cases arise, when k is positive or negative.

Case (*i*). k is $+$ *ve.*

$$\text{Put } \sqrt{k}\, r = r_1$$

$$\text{or } \frac{dr_1}{dr}=\sqrt{k}$$

$$\frac{d^2\eta}{dr^2}=\frac{d}{dr}\left(\frac{d\eta}{dr}\right)=\frac{d}{dr_1}\left(\frac{d\eta}{dr_1}\cdot\frac{dr_1}{dr}\right)\frac{dr_1}{dr}.$$

$$=k.\frac{d^2\eta}{dr_1^2}$$

$$\therefore \quad r^2\frac{d^2\eta}{dr^2}=r_1^2\frac{d^2\eta}{dr_1^2}$$

$$\text{and } r\frac{d\eta}{dr}=r_1\frac{d\eta}{dr_1}$$

Therefore equation (16) is transformed to

$$r_1^2\frac{d^2\eta}{dr_1^2}+r_1\frac{d\eta}{dr_1}+(r_1^2-\mu^2)\eta=0. \qquad . \quad . \quad . \quad . \quad . \quad (17)$$

Now this is clearly Bessel's Equation.

Hence, $\eta = A\,J_\mu + B\,J_{-\mu}$, (2) *when μ is not an integer or zero*

where $J_\mu=\sum\limits_{s=0}^{s=\infty}\frac{(-1)^s}{\Pi(\mu+s)\Pi(s)}\left(\frac{r_1}{2}\right)^{\mu+2s}$ and A and B are arbitrary constants.

When $\mu=0$,

$$\eta=A^1\,J_0+B^1\,Y_0{}^{(3)},$$

where $Y_0=J_0\log r_1+2\,\{J_2-\tfrac{1}{2}J_4+\tfrac{1}{3}J_6-\tfrac{1}{4}J_8+\tfrac{1}{5}J_{10}- \quad . \quad . \quad . \quad . \quad . \quad . \quad\}$,

A$'$, and B$'$ being arbitrary constants.

When μ is a positive integer other than zero,

$$\eta=A''\,J_\mu+B''\,Y_\mu{}^{(4)}$$

where $Y_\mu = J_\mu \log r_1 - \sum_{\nu=1}^{\nu=\infty} (-1)^\nu \frac{\mu+2\nu}{\nu(\mu+\nu)} J_{\mu+2\nu} - \frac{1}{2}\Pi(\mu) \sum_{s=0}^{s=\mu-1} \frac{1}{\mu-s}$

$$\left(\frac{2}{r_1}\right)^{\mu-s} \frac{J_s}{\Pi(s)}$$

Case (ii). k is negative, say, equal to $-k_1$ where k_1 is positive.

Putting $\sqrt{k_1}\, r = r_2$, we get from (16)

$$r_2^{\,2} \frac{d^2\eta}{dr_2^{\,2}} + r_2 \frac{d\eta}{dr_2} - (r_2^{\,2} + \mu^2)\,\eta = 0 \qquad . \quad . \quad . \quad . \qquad (18)$$

whence $\eta = C_1 I_\mu (r_2) + C_2 I_{-\mu}(r_2)^{(5)}$, where $I_\mu (r_2) = \sum_{\nu=0}^{\infty} \frac{(\frac{1}{2}r_2)^{\mu+}}{\nu!(\mu+\nu)!}$

C_1 and C_2 being arbitrary constants, when μ is not an integer or zero.

When μ is an integer or zero.

$\eta = C'_1 I_\mu (r_2) + C'_2 K_\mu (r_2)^{(6)}$, C' and C'_2 being arbitrary constants.

$$\text{where } K_\mu (r_2) = -\sum_{v=0}^{\infty} \frac{(\frac{1}{2}r_2)^{\mu+2\nu}}{\nu!(\mu+\nu)!} \left\{ \log\frac{r_2}{2} + \gamma - \tfrac{1}{2}\sum_{s=1}^{\mu+\nu} s^{-1} - \tfrac{1}{2}\sum_{s=1}^{\nu} s^{-1} \right\}$$

$$+ \tfrac{1}{2}\sum_{=0}^{\mu-1} (\tfrac{1}{2} r_2)^{-\mu+2\nu} \frac{(-1)^{\mu-\nu}(\mu-\nu-1)!}{\nu!}, \qquad \gamma \text{ being Euler's constant.}$$

$\therefore$ the complete solution is given by

$$\xi = e^{\lambda t}\xi_1 = e^{\lambda t}\xi_2 R = e^{\lambda t}.\cos\mu\theta.\ \eta R$$
$$= e^{\lambda t}.\cos\mu\theta\,(R_1 + R_2).\eta.$$

II. $\qquad kr\nabla^2\xi = \dfrac{\partial\xi}{\partial t}, \qquad$ where k is a finite constant.

Put $\xi = e^{-ka^2 t}\, r.u$ . . . . (1)

where α is a finite constant and u stands for a function of r, θ and ϕ only.

Now $\nabla^2 \equiv \dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}$

Substituting the value of ξ in the differential equation, we get

$$e^{-k\alpha^2 t}(-k\alpha^2).ru = e^{-k\alpha^2 t} k\, r \left[r\frac{\partial^2 u}{\partial r^2} + 4\frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]$$

$$\text{or } r^2\frac{\partial^2 u}{\partial r^2}+4r\frac{\partial u}{\partial r}+\alpha^2 ru+2u+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0 \quad . \quad . \quad (2)$$

$$\text{Put } u=\text{R. S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where R is a function of r alone and S, a function of θ and ϕ only so that we get

$$r^2\text{S}\frac{\partial^2\text{R}}{\partial r^2}+4\,r\text{S}\frac{\partial\text{R}}{\partial r}+(\alpha^2 r+2)\ \text{RS}+\frac{\text{R}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\text{S}}{\partial\theta}\right)+\frac{\text{R}}{\sin^2\theta}\frac{\partial^2\text{S}}{\partial\phi^2}=0$$

or dividing by RS,

$$\frac{r^2}{\text{R}}\frac{\partial^2\text{R}}{\partial r^2}+\frac{4r}{\text{R}}\frac{\partial\text{R}}{\partial r}+(\alpha^2 r+2)+\frac{1}{\text{S}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\text{S}}{\partial\theta}\right)+\frac{1}{\text{S}\sin^2\theta}\frac{\partial^2\text{S}}{\partial\phi^2}=0.\ .\ .\ .\ (3)$$

Now (3) can be satisfied if we put

$$\frac{1}{\text{S}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\text{S}}{\partial\theta}\right)+\frac{1}{\text{S}\sin^2\theta}\frac{\partial^2\text{S}}{\partial\phi^2}+n(n+1).=0. \quad . \quad (4)$$

$$\text{and } \frac{r^2}{\text{R}}\frac{\partial^2\text{R}}{\partial r^2}+\frac{4r}{\text{R}}\frac{\partial\text{R}}{\partial r}+(\alpha^2 r+2)=n(n+1)$$

$$\text{i.e., } r^2\frac{d^2\text{R}}{dr^2}+4r\frac{d\text{R}}{dr}+\{\alpha^2 r+2-n(n+1)\}\ \text{R}=0\ . \quad . \quad . \quad (5)$$

n being a positive integer or zero.

The solutions of (4) are clearly Tesseral Harmonics[7].

$$\text{Taking equation (5) we suppose } \text{R}=\sum_{p=0}^{\infty}\text{A}_p\, r^{m+p} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting in (5) we get

$$\text{A}_0[\{m(m-1)+4m+2-n(n+1)\}r^m+\alpha^2 r^{m+1}]+ \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+\text{A}_{p-1}\ [\ldots]+\text{A}_p\ [\ .\ .]+\ldots\ldots\ldots\ldots=0$$

$$\text{or } \text{A}_0\left[\{m^2+3m+2-n(n+1)\}\ r^m+\alpha^2 r^{m+1}\right]+\text{A}_1\Big[\{(m+1)^2+3(m+1)+2$$

$$-n(n+1)\}\ r^{m+1}+\alpha^2 r^{m+2}\Big]+\ \ldots\ \ldots\ \ldots\ +\ \ldots\ldots$$

$$+\ \text{A}_{p-1}\left[\{(m+p-1)^2+3\ (m+p-1)+2-n(n+1)\}r^{m+p-1}+\alpha^2 r^{m+p}\right]$$

$$+\ \text{A}_p\ \left[\{(m+p)^2+3(m+p)+2-n(n+1)\}r^{m+p}+\alpha^2 r^{m+p+1}\right]+\cdots=0.$$

Comparing the co-efficients we get

$$m^2+3m+2-n(n+1)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{and } \left[(m+p)^2+3(m+p)+2-n(n+1)\right]\ \text{A}_p+\alpha^2\text{A}_{p-1}=0 \quad . \quad . \quad . \quad (8)$$

Taking (7) we get

$$m^2+3m+2-n(n+1)=0$$

$$\text{or } \{m+(n+2)\}\{m-(n-1)\}=0$$

$$\therefore\ m=-(n+2) \text{ or } (n-1)$$

$$\text{say, } m_1=n-1 \text{ and } m_2=-(n+2).$$

Again, from (8)

$$A_p=-\frac{\alpha^2}{(m+p^2)+3(m+p)+2-n(n+1)}A_{p-1} \quad . \quad . \quad . \quad (10)$$

for $p\geq 1$

(10) shows that the series thus found will be absolutely and uniformly convergent for all finite values of r including those in the neighbourhood of $r=0$, which is a pole in this case. For values of r in the neighbourhood of $r=0$, R will remain finite when we reject the negative value of m and accept that one which is not negative, i.e., we must take the value of m to be $n-1$.

It is evident from (10) that all the A's can be expressed in terms of a single arbitrary constant A_0.

Now in (10)

Put $p=1$, $A_1=-\dfrac{\alpha^2}{(m+1)^2+3(m+1)+2-n(n+1)}A_0$

$p=2$, $A_2=-\dfrac{\alpha^2}{(m+2)^2+3(m+2)+2-n(n+1)}A_1$

$$=(-1)^2\frac{\alpha^4}{\{(m+1)^2+3(m+1)+2-n(n+1)\}\ \{(m+2)^2+3(m+2)+2-n(n+1)\}}A_0.$$

& so on.

Hence,

$$R=\ Ar^m\Big[=1-\frac{\alpha^2}{(m+1)^2+3(m+1)+2-n(n+1)}r$$

$$+\frac{\alpha^4}{\{(m+2)^2+3(m+2)+2-n(n+1)\}\{(m+1)^2+3(m+1)+2-n(n+1)\}}r^2\Big]$$

$$+\ \ldots\ \ldots\ \ldots\ \ldots$$

where A is an arbitrary constant.

Therefore we get the complete solution of II to be

$$\xi = e^{-k\alpha^2 t}\, r.\, u$$

$$= e^{-k\alpha^2 t}\, r.\, \text{R. S.}$$

$$= e^{-k\alpha^2 t}\, r\, \text{S}\, (\text{R}_1+\text{R}_2)$$

where $R_1=[R]_{m=n-1}$ and $\left[\dfrac{\partial R}{\partial m}\right]_{m=-(n+2)}=R_2$ by suitably modifying the arbitrary constant, by the method of Frobenius, since the difference between the two values of m is an integer.

III. $kr^2\nabla^2\xi = \dfrac{\partial\xi}{\partial t}$, k being a finite constant.

Here put $\xi = e^{-ka^2t}r^2u$ where u is a function of r, θ and ϕ, and α is a finite constant, so that we have

$$\text{since } \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},$$

$$(-k\alpha^2)e^{-k\alpha^2t}r.^2u = e^{-k\alpha^2t}kr^2\left[r^2\frac{\partial^2u}{\partial r^2} + 4r\frac{\partial u}{\partial r} + 2u + 2r\frac{\partial u}{\partial r} + 4u + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}\right]$$

$$\text{or } r^2\frac{\partial^2u}{\partial r^2} + 6r\frac{\partial u}{\partial r} + (\alpha^2+6)u + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2u}{\partial\phi^2} = 0$$

Put u = R.V., where R is a function of r alone and V that of θ and ϕ only. Then,

$$r^2V\frac{\partial^2R}{\partial r^2} + 6rV\frac{\partial R}{\partial r} + (\alpha^2+6)\,RV + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2V}{\partial\phi^2} = 0$$

Dividing by RV, we get

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{6r}{R}\frac{dR}{dr} + (\alpha^2+6) + \frac{1}{V\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{V\sin^2\theta}\frac{\partial^2V}{\partial\phi^2} = 0$$

Now this equation can be satisfied by taking

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{6r}{R}\frac{dR}{dr} + (\alpha^2+6) = n(n+1) \quad . \quad . \quad . \quad (1)$$

$$\text{and } \frac{1}{V\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{V\sin^2\theta}\frac{\partial^2V}{\partial\phi^2} + n(n+1) = 0 \quad . \quad (2)$$

n being a positive integer or zero.

The solution of (2), *viz.*, V, is evidently Tesseral Harmonics[7]
Taking (1) we get

$$r^2\frac{d^2R}{dr^2} + 6r\frac{dR}{dr} + (\alpha^2+6-n^2-n)R = 0$$

This is clearly a homogeneous equation of the second order.
For its complementary function we have

$$D(D-1) + 6D + m = 0. \qquad \text{where 'D' has its usual significance and } m = \alpha^2+6-n^2-n.$$

$$\text{or } D^2+5D+m = 0$$

$$\therefore\ D = \frac{-5\pm\sqrt{25-4m}}{2} = \beta_1 \text{ or } \beta_2 \text{ (say)}.$$

$\therefore\ \mathrm{R} = c_1 r^{\beta_1} + c_2 r^{\beta_2}$, where c_1 and c_2 are arbitrary constants.

$$\therefore\ \xi = e^{-ka^2t} r^2 u$$
$$= e^{-ka^2t} r^2\, \mathrm{RV}. = e^{ka^2t} \mathrm{V} r^2 (c_1 r^{\beta_1} + c_2 r^{\beta_2})$$
$$= e^{-ka^2t} \mathrm{V}\, (c_1 r^{\beta_1+2} + c_2 r^{\beta_2+2})$$

IV. $\dfrac{\partial\xi}{\partial t} = (c+kr)\nabla^2\xi,$ where c and k are finite constants different from zero.

Put $\xi = e^{-ka^2t}(c+kr).\ u$, where u is a function of r, θ and ϕ and α is a finite constant.

Now $\nabla^2 \equiv \dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}$

Substituting the value of ξ, we get

$$(e^{-ka^2t})(-ka^2)(c+kr)u = (c+kr)e^{-ka^2t}\left[(c+kr)\frac{\partial^2 u}{\partial r^2} + 2k\frac{\partial u}{\partial r} + \frac{2ku}{r} + \frac{2(c+kr)}{r}\frac{\partial u}{\partial r} + \frac{c+kr}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{c+kr}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]$$

$$\text{or } -ka^2 u = \left[(c+kr)\frac{\partial^2 u}{\partial r^2} + 2\frac{c+2kr}{r}\frac{\partial u}{\partial r} + \frac{2ku}{r} + \frac{c+kr}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{c+kr}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]$$

Put $u = \mathrm{R.S.}$, where R is a function of r only and S that of θ and ϕ only.
so that

$$-ka^2\mathrm{RS} = \left[(c+kr)\mathrm{S}\frac{\partial^2\mathrm{R}}{\partial r^2} + 2.\frac{c+2kr}{r}.\mathrm{S}.\frac{\partial\mathrm{R}}{\partial r} + 2\frac{k}{r}\mathrm{RS} + \frac{c+kr}{r^2\sin\theta}\mathrm{R}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{S}}{\partial\theta}\right) + \frac{c+kr}{r^2\sin^2\theta}\mathrm{R}\frac{\partial^2\mathrm{S}}{\partial\phi^2}\right]$$

Dividing both sides by RS, we get

$$-ka^2 = \left[\frac{c+kr}{\mathrm{R}}\frac{\partial^2\mathrm{R}}{\partial r^2} + \frac{2}{\mathrm{R}}.\frac{c+2kr}{r}\frac{\partial\mathrm{R}}{\partial r} + \frac{2\mathrm{K}}{r} + \frac{c+kr}{r^2\sin\theta}\frac{1}{\mathrm{S}}.\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{S}}{\partial\theta}\right) + \frac{c+kr}{r^2\sin^2\theta}\frac{1}{\mathrm{S}}\frac{\partial^2\mathrm{S}}{\partial\phi^2}\right]$$

Now multiplying both sides by $\dfrac{r^2}{c+kr}$ and transposing, we get

$$\frac{r^2}{\mathrm{R}}\frac{\partial^2\mathrm{R}}{\partial r^2} + 2\frac{r(c+2kr)}{c+kr}\frac{1}{\mathrm{R}}\frac{\partial\mathrm{R}}{\partial r} + \frac{2kr}{c+kr} + \frac{ka^2r^2}{c+kr} + \frac{1}{\mathrm{S}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{S}}{\partial\theta}\right) + \frac{1}{\mathrm{S}\sin^2\theta}\frac{\partial^2\mathrm{S}}{\partial\phi^2} = 0 \quad \ldots (1)$$

Equation(1) can be satisfied if we write

$$\left.\begin{aligned} &\frac{r^2}{\mathrm{R}}\frac{d^2\mathrm{R}}{dr^2} + \frac{2r}{\mathrm{R}}.\frac{c+2kr}{c+kr}\frac{d\mathrm{R}}{dr} + \frac{2kr}{c+kr} + \frac{\mathrm{K}a^2r^2}{c+kr} = n(n+1) \quad \ldots\ldots (2) \\ \text{and } &\frac{1}{\mathrm{S}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{S}}{\partial\theta}\right) + \frac{1}{\mathrm{S}\sin^2\theta}\frac{\partial^2\mathrm{S}}{\partial\phi^2} + n(n+1) = 0 \quad \ldots\ldots\ldots (3) \end{aligned}\right\} \begin{array}{l} n \text{ being } a + ve \\ \text{integer or} \\ \text{zero.} \end{array}$$

The solutions of (3) will be evidently Tesseral Harmonics[7].

Now let us come back to Equation(2). Simplifying, we get

$$r^2(c+kr)\frac{d^2R}{dr^2}+2r(c+2kr)\frac{dR}{dr}+R\{2kr+K\alpha^2r^2-n(n+1)\ (c+kr)\}=0 \quad . \quad . \quad . \quad (4)$$

In this case we find that there are two poles, viz., at $r=0$ and $c+kr=0$. Again $r=0$, is a double pole. It is quite clear that if we want to solve (4) in a series, we must have different series in the neighbourhood of $r=0$ and $c+kr=0$.

Case (*i*). Simplifying (4), further we get

$$r^2(c+kr)\frac{d^2R}{dr^2}+2r(c+2kr)\frac{dR}{dr}+R\{k\alpha^2r^2-kr(n+2)\ (n-1)-cn(n+1)\}=0 \quad . \quad . \quad . \quad (4i)$$

Suppose $R=\sum_{p=0}^{\infty} A_p\, r^{m+p}$.

$$=A_0r^m+A_1r^{m+1}+\ \ldots\ldots\ +A_pr^{m+p}+\ \ldots\ldots$$

satisfies (4*i*).

Substituting we get

$$A_0[r^2(c+kr)m(m-1)r^{m-2}+2r(c+2kr)mr^{m-1}+r^m\ \{k\alpha^2r^2-kr(n+2)\ (n-1)-cn(n+1)\}]+\ \ldots\ldots\ =0$$

or $$A_0[cr^m\{m^2+m-n(n+1)\}+kr^{m+1}\{m(m-1)+4m-(n+2)\ (n-1)\}+k\alpha^2r^{m+2}]+\ \ldots\ldots\ =0.$$

or $$\begin{aligned}&A_0[cr^m\{m^2+m-n(n+1)\}+kr^{m+1}\{m^2+3m-(n+2)\ (n-1)\}+k\alpha^2r^{m+2}]\\&+A_1[cr^{m+1}\{(m+1)^2+(m+1)-n(n+1)\}+kr^{m+2}\{(m+1)^2+3(m+1)-\\&\qquad(n+2)\ (n-1)\}+k\alpha^2r^{m+3}]\\&+\ \ldots\ldots\ +\ \ldots\ldots\ +\ \ldots\ldots\\&+\ \ldots\ +\ \ldots\ +\ \ldots\ +\ \ldots\\&+A_{p-1}[cr^{m+p-1}\{(m+p-1)^2+(m+p-1)-n(n+1)\}+kr^{m+p}\{(m+p-1)^2\\&\qquad+3(m+p-1)-(n+2)\ (n-1)\}+k\alpha^2r^{m+p+1}]\\&+A_p[cr^{m+p}\{(m+p)^2+(m+p)-n(n+1)\}+kr^{m+p+1}\{(m+p)^2+3(m+p)\\&\qquad-(n+2)\ (n-1)\}+k\alpha^2r^{m+p+2}]\\&+A_{p+1}[cr^{m+p+1}\{(m+p+1)^2+(m+p+1)-n(n+1)\}+kr^{m+p+2}\{(m+p+1)^2\\&\qquad+3(m+p+1)-(n+2)\ (n-1)\}+k\alpha^2r^{m+p+3}]+\ \ldots\ldots\ 0\end{aligned}$$

Comparing the coefficients, we get

$$m^2+m-n(n+1)=0 \text{ or } (m+n+1)\ (m-n)=0 \quad \therefore m=n \text{ or } -(n+1) \quad . \quad . \quad (5)$$

$$A_0k\{m^2+3m-(n+2)(n-1)\}+A_1c\{(m+1)^2+(m+1)-n(n+1)\}=0 \quad \dots \quad (6)$$

$$A_{p-1}k\alpha^2+A_p.k.\{(m+p)^2+3(m+p)-(n+2)(n-1)\}+A_{p+1}c\{(m+p+1)^2+(m+p+1)-n(n+1)\}=0 \quad \dots \quad (7)$$

Now from (7)

$$\frac{k\alpha^2}{(m+p)^2+3(m+p)-(n+2)(n-1)}+\frac{A_p}{A_{p+1}}k+\frac{A_{p-1}}{A_{p-1}}\cdot c\cdot\frac{(m+p+1)^2+(m+p+1)-n(n+1)}{(m+p)^2+3(m+p)-(n+2)(n-1)}=0.$$

Taking limits as $p\to\infty$, we find that $Lt\,\dfrac{A_p}{A_{p-1}}\,k+Lt\,\dfrac{A_{p+1}}{A_{p+1}}\,c=0$

whereby $Lt\,\dfrac{A_{p+1}}{A_p}\,r=-\dfrac{kr}{c}.$

This shows that $Lt\left|\dfrac{A_{p+1}}{A_p}\,r\right|=\left|\dfrac{kr}{c}\right|$. This indicates that the series thus found will be absolutely convergent provided $r<\dfrac{|c|}{|k|}$ and simply convergent when $-\dfrac{kr}{c}<1$ or $-kr<c$ or $r>-\dfrac{c}{k}$.

For values of R in the neighbourhood of zero, the expression R will remain finite provided we take the positive value of m, viz., $m=n$ from (5).

In order to find the nature of R when $c+kr=0$, we shall have to get the value of R in powers of $c+kr$.

Thus we get two series and this gives the complete solution of R.

The circle of convergence is of radius $\left|\dfrac{c}{k}\right|$, where the point $r=-\dfrac{c}{k}$ is a branch point[8] of the solution, the circle having its centre at $r=0$.

The solutions will be $[R]_{m=n,}$ and $\left[\dfrac{\partial R}{\partial m}\right]_{m=-(n+1)}$ with suitable modification of the arbitrary constant.

Case (ii). $$r^2(c+kr)\frac{d^2R}{dr^2}+2r(c+2kr)\frac{dR}{dr}+R\{k\alpha^2r^2-kr(n+2)(n-1)-cn(n+1)\}=0 \quad \dots\dots \quad (4i)$$

$$\text{Put } c+kr=v. \text{ or } \frac{dv}{dr}=k$$

$$\frac{d^2R}{dr^2}=k^2\frac{d^2R}{dv^2} \text{ and } \frac{dR}{dr}=k.\frac{dR}{dr}.$$

so that we get

$$(v-c)^2v\frac{d^2R}{dv^2}+2(v-c)(2v-c)\frac{dR}{dv}+R\left\{\frac{(v-c)^2\alpha^2}{k}-(v-c)(n+2)(n-1)-cn(n+1)\right\}=0$$

$$\text{or } kv\,(v-c)^2\frac{d^2R}{dv^2}+2k\,(v-c)\,(2v-c)\frac{dR}{dv}+R\,\{\,(v-c)^2\alpha^2-(v-c)\,k\,(n+2)(n-1)-ckn\,(n+1)\,\}=0$$

$$\text{or } kv\,(v^2-2vc+c^2)\frac{d^2R}{dv^2}+2k\,(2v^2-3vc+c^2)\frac{dR}{dv}+R\,\{v^2\alpha^2-2vc\alpha^2-vk\,(n+2)\,(n-1)+c^2\alpha^2+ck\,(n+2)\,(n-1)-ckn\,(n+1)\}=0$$

$$\text{or } k\,(v^3-2v^2c+c^2v)\frac{d^2R}{dv^2}+2k\,(2v^2-3vc+c^2)\frac{dR}{dv}+R\,[v^2\alpha^2-v\,\{2c\alpha^2+c^2\alpha^2+k\,(n+2)\,(n-1)\}-2ck]=0 \quad . \quad . \quad (8)$$

Suppose $R=\sum_{p=1}^{\infty} B_p\, v m_p$ satisfies (8).

Substituting we get

$$B_1\Big[k\,(v^3-2v^2c+c^2v)\,m_1\,(m_1-1)\,v^{m_1-2}+2k\,(2v^2-3vc+c^2)\,m_1v^{m_1-1}+v^{m_1}\Big\{v^2\alpha^2-v\Big(2c\alpha^2+k.\overline{n+2}.\overline{n-1}.+c^2\alpha^2\Big)-2ck\Big\}\Big]$$

$$+\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad =0$$

$$\text{or } B_1\Big[v^{m_1-1}\Big\{kc^2m_1\,(m_1-1)+2kc^2m_1\Big\}+v^{m_1}\Big\{-2kcm_1\,(m_1-1)-6kcm_1-2ck-2ck+c^2\alpha^2\Big\}+v^{m_1+1}\Big\{km_1\,(m_1-1)+4km_1-2c\alpha^2-k\,(n+2)(n-1)\Big\}+\alpha^2v^{m_1+2}\Big]$$

$$+\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad =0$$

$$\text{or } B_1\,[kc^2v^{m_1-1}m_1\,(m_1+1)-2ckvm_1\,(m_1+1)^2+v^{m_1}\,(c^2\alpha^2)+v^{m_1+1}\,\{km_1\,(m_1+3)-2c\alpha^2-k\,(n+2)\,(n-1)\}+\alpha^2v^{m_1+2}]$$

$$+B_2\,[kc^2v^{m_2-1}m_2\,(m_2+1)-2ckv^{m_2}\,(m_2+1)^2+v^{m_2}\,(c^2\alpha^2)+v^{m_2+1}\,\{km_2\,(m_2+3)-2c\alpha^2-k\,(n+2)\,(n-1)\}+\alpha^2v^{m_2+2}]$$

$$+B_3\,[kc^2v^{m_3-1}m_3\,(m_3+1)-2ckv^{m_3}\,(m_3+1)^2+v^{m_3}\,(c^2\alpha^2)+v^{m_3+1}\{km_3\,(m_3+3)-2c\alpha^2-k\,(n+2)\,(n-1)\}+\alpha^2v^{m_3+2}]$$

. . . .

. . . .

$$+B_{p-1}\,[kc^2v^{m_{p-1}-1}\,m_{p-1}(m_{p-1}+1)-2ckv^{m_{p-1}}(m_{p-1}+1)^2+v^{m_{p-1}}(c^2\alpha^2)$$
$$+v^{m_{p-1}+1}\{km_{p-1}(m_{p-1}+3)-2c\alpha^2-k\,(n+2)\,(n-1)\}+\alpha^2v^{m_{p-1}+2}$$

$$+B_p\,[kc^2v^{m_p-1}\,m_p\,(m_p+1)-2ckv^{m_p}(m_p+1)^2+v^{m_p}(c^2\alpha^2)+v^{m_p+1}\{km_p\,(m_p+3)$$
$$-2c\alpha^2-k(n+2)\,(n-1)\}+\alpha^2v^{m_p+2}]$$

$$+B_{p+1}\,[kc^2v^{m_{p+1}-1}m_{p+1}(m_{p+1}+1)-2ckv^{m_{p+1}}(m_{p+1}+1)^2+v^{m_{p+1}}(c^2\alpha^2)$$
$$+v^{m_{p+1}+1}\{km_{p+1}(m_{p+1}+3)-2c\alpha^2-k\,(n+2)\,(n-1)\}+\alpha^2v^{m_{p+1}+2}]$$

$$+B_{p+2}\,[kc^2v^{m_{p+2}}\,m_{p+2}\,(m_{p+2}+1)-2ckv^{m_{p+2}}(m_{p+2}+1)^2+v^{m_{p+2}}(c^2\alpha^2)$$
$$+v^{m_{p+2}+1}\{km^{p+2}m_{p+2}+3)\;-2c\alpha^2-k\;(n+2)\,(n-1)\}+\alpha^2v^{m_{p+2}+2}]$$
$$+\quad.\quad.\quad.\quad.\quad.\quad.\quad=0$$

Comparing the terms we get

$$m_1\,(m_1+1)=0\quad\therefore m_1=0\text{ or}-1\qquad\qquad(9)$$

$m_3=m_1+1_1$, $m_3=m_2+1$, etc.

$$\text{so that } m_p=m_1+(p-1)\qquad\qquad(10)$$

$$\{c^2\alpha^2-2ck(m_1+1)^2\}\,B_1+kc^2m_2\,(m_2+1)\,B_2=0\qquad\qquad(11)$$

$$B_1\,\{km_1\,(m_1+3)\;-2c\alpha^2-k\,(n+2)\,(n-1)\}\;-2ck\,(m_2+1)^2B_2+c^2\alpha^2B_2$$
$$+kc^2\,m_3\,(m_3+1)\,B_3=0\qquad\qquad(12)$$

$$\alpha^2B_{p-1}+B_p\;\{km_p\,(m_p+3)-2c\alpha^2-k\,(n+2)\,(n-1)\}\;-2ck\;(m_{p+1}+1)^2B_{p+1}$$
$$+c^2\alpha^2B_{p+1}+kc^2m_{p+2}\,(m_{p+2}+1)\,B_{p+2}=0.\qquad\qquad(13)$$

Simplifying we get

$$\{c^2\alpha^2-2ck\,(m_1+1)^2\}\,B_1+kc^2\,(m_1+1)\,(m_1+2)\,B_2=0\qquad\qquad(11i)$$

$$B_1\{km_1\,(m_1+3)\;-2c\alpha^2-k\,(n+2)\,(n-1)\}\;-2ck\;(m_1+2)^2B_2$$
$$+c^2\alpha^2B_2+kc^2\,(m_1+2)\,(m_1+3)\,B_3=0\qquad\qquad(12i)$$

$$\alpha^2B_{p-1}+B_p\;\{k\,(m_1+p-1)\,(m_1+p+2)\;-2c\alpha^2-k\,(n+2)\,(n-1)\}$$
$$+c^2\alpha^2B_{p+1}-2ck\,(m_1+p+1)^2B_{p+1}+kc^2\,(m_1+p+1)\,(m_1+p+2)\,B_{p+2}=0\qquad(13i)$$

From (13*i*) we get

$$\alpha^2+\frac{B_p}{B_{p-1}}\;\{k\,(m_1+p-1)\,(m_1+p+2)-2c\alpha^2-k\,(n+2)\,(n-1)\}$$

$$-\{2ck\,(m_1+p+1)^2-c^2\alpha^2\}\;\frac{B_{p+1}}{B_{p-1}}+kc^2(m_1+p+1)\,(m_1+p+2)\frac{B_{p+2}}{B_{p-1}}=0$$

or $\quad \alpha^2+\dfrac{B_p}{B_{p-1}}\, k0_1(p^2)-2ck0_2(p^2)\,\dfrac{B_{p+1}}{B_{p-1}}+kc^2 0_3(p^2).\,\dfrac{B_{p+2}}{B_{p-1}}=0$

or $$\frac{\alpha^2}{0_1\,(p^2)}+k.\,\frac{B_p}{B_{p-1}}-2ck\,\frac{0_2\,(p^2)}{0_1\,(p^2)}\frac{B_{p+1}}{B_{p-1}}+kc^2.\,\frac{0_3\,(p^2)}{0_1\,(p^2)}\frac{B_{p+2}}{B_{p-1}}=0 \quad . \quad . \qquad (14)$$

Let $$\underset{p\to\infty}{\text{Lt}}\ \frac{B_p}{B_{p-1}}=k_1$$

Where by
$$\underset{p\to\infty}{\text{Lt}}\ \frac{B_{p+1}}{B_{p-1}}=k_1{}^2$$

and $$\underset{p\to\infty}{\text{Lt}}\ \frac{B_{p+2}}{B_{p-1}}=k_1{}^3$$

Taking limits in (14) as $p\to\infty$, we get
$$k_1\ k-2ck\ k_1{}^2+kc^2\ k_1{}^3=0$$

or $$k_1k\ (k_1{}^2c^2-2ck_1+1)=0$$

This shows that either $k_1=0$ or $\dfrac{1}{c}$,

i.e. $$\text{Lt}\ \frac{B_p}{B_{p-1}}v=\frac{v}{c} \text{ or zero.}$$

When the limit is zero the series is convergent for all values of v but $v=c$ is a singular point, so unless we prove the convergence of the series at $v=c$, we cannot take k_1 to be zero.

$\therefore$ If $v<c$, i.e., when $kr<o$, the series is Convergent

and if $v>c$, i.e., $kr<0$, the series is Divergent.

When $v=c$ or $r=0$ we again go back to our Case (i).

Here also the circle of convergence may be drawn with centre $-\dfrac{c}{k}$ and radius $\left|\dfrac{c}{k}\right|$ where $r=0$ is a branch point of the solution.

The solutions will be $[R]_{m_1=0}$ and $\left[\dfrac{\partial R}{\partial m_1}\right]_{m_1=-1}$, modifying the arbitrary constant suitably.

We have thus obtained a fundamental system of solutions valid in the vicinity of regular singular points. (The points $r=0$ and $c+kr=0$ are, of course, regular singular points.) Again, if we assume that there is a domain S in which the coefficients of $\dfrac{dR}{dr}$ and R are analytic except at the two poles $r=0$ and $c+kr=0$. (The coefficient of $\left(\dfrac{d^2R}{dr^2}\right)$ being taken as unity) we see that the two singularities in S of the

equation are regular points, each member of a pair of fundamental solutions is analytic at all points of S except at the singularities of the equation, which are branch points of the solution.

V. $(a_0+b_0 x)\nabla^2\zeta+c_0 \dfrac{\partial\zeta}{\partial x}=0$ where a_0, b_0 and c_0 are finite constants.

This can be put in the form

$$(a+bx)\nabla^2\zeta+\frac{\partial\zeta}{\partial x}=0$$

or

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+\frac{\partial^2\zeta}{\partial z^2}+\frac{1}{a+bx}\frac{\partial\zeta}{\partial x}=0$$

Put $\zeta=R\zeta_1$, where R is a function of x alone and ζ_1 that of y and z only.

Substituting we get

$$\zeta_1\frac{\partial^2 R}{\partial x^2}+R\left(\frac{\partial^2\zeta_1}{\partial y^2}+\frac{\partial^2\zeta_1}{\partial z^2}\right)+\frac{1}{a+bx}\zeta_1\frac{\partial R}{\partial x}=0$$

or

$$\frac{1}{R}\left(\frac{\partial^2 R}{\partial x^2}+\frac{1}{a+bx}\frac{\partial R}{\partial x}\right)+\frac{1}{\zeta_1}\left(\frac{\partial^2\zeta_1}{\partial y^2}+\frac{\partial^2\zeta_1}{\partial x^2}\right)=0$$

This equation can be satisfied if we write

$$\frac{1}{R}\frac{\partial^2 R}{\partial x^2}+\frac{1}{a+bx}\frac{1}{R}\frac{\partial R}{\partial x}=k \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\frac{1}{\zeta_1}\left(\frac{\partial^2\zeta_1}{\partial y^2}+\frac{\partial^2\zeta_1}{\partial z^2}\right)+k=0 \quad \text{or} \quad \frac{\partial^2\zeta_1}{\partial y^2}+\frac{\partial^2\zeta_1}{\partial z^2}+k\zeta_1=0. \quad . \quad (2)$$

k being a finite constant other than zero.

Now equation (2) is the same as equation (8) in I and its solution has been exhibited there.

Turning to equation (1) and simplifying, we get

$$(a+bx)\frac{d^2R}{dx}+\frac{dR}{dx}-k(a+bx)R=0 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Put $a+bx=u$, so that

$$b^2u\frac{d^2R}{du^2}+b\frac{dR}{du}-kuR=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let

$$R=\sum_{p=1}^{\infty}A_p u^{m_p}\equiv A_1u^{m_1}+A_2u^{m_2}+\ .\ .\ .+A_p u^{m_p}+\ .\ .\ .$$

satisfy (3), so that by substituting the value of R in (3) we get

$$A_1\left[b^2u\,m_1(m_1-1)u^{m_1-2}+bm_1u^{m_1-1}-ku^{m_1+1}\right]+\ .\ .\ .\ .\ .\ \equiv 0$$

or

$$A_1\left[u^{m_1-1}(b^2m_1^2-b^2m_1+bm_1)-ku^{m_1+1}\right]+\ .\ .\ .\ .\ .\ \equiv 0$$

or $$A_1[bm_1(bm_1-b+1)u^{m_1-1}-ku^{m_1+1}]+A_2[bm_2(bm_2-b+1)u^{m_2-1}-ku^{m_2+1}]+\ldots\ldots+A_{p-1}[bm_{p-1}(bm_{p-1}-b+1)u^{m_{p-1}-1}-ku^{m_{p-1}+1}]+A_p[bm_p(bm_p-b+1)\,u^{m_p-1}-ku^{m_p+1}]+\ldots\ldots\equiv 0$$

Comparing the terms we get

$$bm_1(bm_1-b+1)=0,\ i.e.,\ m_1=0 \text{ or } \frac{b-1}{b} \qquad (4)$$

$$m_2-1=m_1+1 \text{ or } m_2=m_1+2$$

$$m_3=m_2+2=m_1+4 \text{ etc.}$$

so that $$m_p=m_1+2(p-1) \qquad (5)$$

Again, $$-A_{p-1}k+bm_p(bm_p-b+1)A_p=0$$

or $$\frac{A_p}{A_{p-1}}=\frac{k}{bm_p(bm_p-b+1)}$$

$$=\frac{k}{b(m_1+2p-2)\,(bm_1+2bp-2b-b+1)},\ \text{for } p\geq 2$$

or $$\frac{A_pu^2}{A_{p-1}}=\frac{ku^2}{b(m_1+2p-2)\,(bm_1+2bp-3b+1)}$$

Thus all the A's can be expressed in terms of a single arbitrary constant.

Again, $\frac{A_pu^2}{A_{p-1}}\to 0$ as $p\to\infty$ and thus independent of b or m_1.

Here the series thus found is absolutely and uniformly Convergent for all values of u, including those in the neighbourhood of $u=0$.

When $b>1$, and $\frac{b-1}{b}$ is $+ve$ the above statement holds for all values of u but if $b<1$, such that $\frac{b-1}{b}$ is negative, we must take $m_1=0$ in order that R may be finite in the neighbourhood of $u=0$.

Again, when $b=1$, $m_1=0$, thus the two series coincide into one. In that case we shall replace m_1 by α and then

$$\frac{A_p}{A_{p-1}}=\frac{k}{b(\alpha+2p-2)\,(b\alpha+2bp-3b+1)}$$

Then the complete solution will be

$$\Big[R\Big]_{\alpha=0}, \text{ and } \left[\frac{\partial R}{\partial \alpha}\right]_{\alpha=0} \quad \text{by the method of Frobenius}^{9}$$

When $\frac{b-1}{b}$ is an integer$=s$ the solutions are

$$\Big[R\Big]_{\alpha=s},\ \left[\frac{\partial R}{\partial \alpha}\right]_{\alpha=0} \text{ with suitable modification of arbitrary constants.}$$

Now $A_p = \frac{k}{b}\frac{1}{(m_1+2p-2)(bm_1+2bp-3b+1)}A_{p-1}$ for $p \geq 2$

Case (*i*). $m_1=0$

$$A_p = \frac{k}{2b}\frac{1}{(p-1)(2bp-3b+1)}A_{p-1} \qquad \text{for } p\geq 2$$

Put $p=2$, $A_2 = \frac{k}{2b}\frac{1}{b+1}A_1$.

$p=3$, $A_3 = \frac{k}{2b}\frac{1}{2(3b+1)}A_2 = \left(\frac{k}{2b}\right)^2\frac{1}{1\cdot 2\cdot(b+1)(3b+1)}A_1$

$$= \left(\frac{k}{2b}\right)^2\frac{1}{2!}\frac{1}{(b+1)(3b+1)}A_1$$

$p=4$, $A_4 = \frac{k}{2b}\frac{1}{3.(5b+1)}A_3$

$$= \left(\frac{k}{2b}\right)^3\frac{1}{3!(b+1)(3b+1)(5b+1)}A_1$$

$p=5$, $A_5 = \left(\frac{k}{2b}\right)^4\frac{1}{4!}\frac{1}{(b+1)(3b+1)(5b+1)(7b+1)}A_1$

$$\therefore R_1 = A_1\Big[1+\frac{k}{2b}\cdot\frac{1}{b+1}\cdot u^2+\left(\frac{k}{2b}\right)^2\frac{1}{2!}\frac{1}{(b+1)(3b+1)}u^4 + \left(\frac{k}{2b}\right)^3\frac{1}{3!}\frac{1}{(b+1)(3b+1)(5b+1)}u^6 + \left(\frac{k}{2b}\right)^4\frac{1}{4!}\frac{1}{(b+1)(3b+1)(5b+1)(7b+1)}u^8+\ldots\Big]$$

A_1 being an arbitrary constant.

Case (*ii*). $m_1=\frac{b-1}{b}$, or $bm_1=b-1$

$$A_p^1 = \frac{k}{(bm_1+2bp-2b)(bm_1+2bp-3b+1)}A^1_{p-1} \qquad \text{for } p\geq 2$$

$$= \frac{k}{(2bp-b-1)(2bp-2b)}A^1_{p-1}$$

$$= \frac{k}{2b}\cdot\frac{1}{p-1}\cdot\frac{1}{2bp-b-1}A^1_{p-1}$$

$$A^1_2 = \frac{k}{2b}\cdot\frac{1}{1\cdot(3b-1)}A^1_1$$

$$A^1_3 = \frac{k}{2b}\cdot\frac{1}{2\cdot(5b-1)}A^1_2 = \left(\frac{k}{2b}\right)^3\frac{1}{2!(3b-1)(5b-1)}A_1^1$$

$$A^1_4 = \left(\frac{k}{2b}\right)^3\frac{1}{3!}\frac{1}{(3b-1)(5b-1)(7b-1)}A_1^1$$

$$A^1{}_5=\left(\frac{k}{2b}\right)^4\frac{1}{4!}\frac{1}{(3b-1)(5b-1)(7b-1)(9b-1)}A^1{}_1$$

$$\therefore\ R_2 = A^1{}_1\, u^{\frac{b-1}{b}}\left[1+\frac{k}{2b}\frac{1}{3b-1}u^2+\left(\frac{k}{2b}\right)^2\frac{1}{2!}\frac{1}{(3b-1)(5b-1)}u^4\right.$$
$$+\left(\frac{k}{2b}\right)^3\frac{1}{3!}\frac{1}{(3b-1)(5b-1)(7b-1)}u^6$$
$$\left.+\left(\frac{k}{2b}\right)^4\frac{1}{4!}\frac{1}{(3b-1)(5b-1)(7b-1)(9b-1)}u^8+\ldots\ldots\right]$$

A^1 being another arbitrary constant.

Hence, the solution of V is given by

$$\zeta = \zeta_1\,(R_1 + R_2)$$

VI. $$\frac{\partial v}{\partial t}=(a_0+a_1x+a_2y+a_3z)\nabla^2V_1,$$ the a's being finite constants

This can be written as

$$\frac{\partial v}{\partial t}=k(l_0+l_1x+l_2y+l_3z)\nabla^2V$$

where $k=\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}$, $kl_0=a_0$; $kl_1=a_1$; $kl_2=a_2$, and $kl_3=a_3$

From these we get $l_1{}^2+l_2{}^2+l_3{}^2=1$.

Now let us change our axes orthogonally to another set of axes, ξ, η, ζ where the order of transformation is shown herewith.[10]

	ξ	η	ζ
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

The relation between the l's, m's and n's are too well known to be cited here. After transformation we have

$$\frac{\partial v}{\partial t}=k(l_0+\xi)\nabla^2V,\ \text{this } \nabla^2V \text{ meaning } \frac{\partial^2V}{\partial\xi^2}+\frac{\partial^2V}{\partial\eta^2}+\frac{\partial^2V}{\partial y^2}$$
$$=(a_0+k\xi)\nabla^2V$$

The solution of 'V' in terms of ξ, η, and s can be found as we have done in I.

My best thanks are due to Prof. A. C. Banerji whose suggestions have helped me a good deal in finding the solution of the equations, I also thank Mr. H. K. Sen for some suggestions.

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SKY-WAVE TRANSMISSION WITH VARIABLE ANGLE OF RADIATION

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Introduction

The reflection of sky-waves from the ionosphere depends upon its height and density, and the critical frequency at normal incidence at a certain time of the day or night is determined by the effective heights and densities of the reflecting layers at that time. For oblique incidence which is the case in actual transmission, the angle at which this frequency is being radiated from the transmitter, *i.e.*, the angle of radiation, is a very important factor in determining the condition of reflection.

The heights and densities of the reflecting layers in the ionosphere change from time to time, hence the maximum usable frequency also has to be changed to suit these varying conditions. Similarly, it is possible to make changes in the angle of radiation so that the wave will still be reflected to the same place though the heights and densities of the reflecting layers have changed.

In the broadcast frequency range, the use of concentric arrays of short antennas to provide control over vertical directivity, was investigated with interesting results.[1] Arrays of this kind have been shown to provide somewhat greater gain than can be obtained from a single vertical half-wave antenna and by arrangements of the concentric groups, it becomes possible to control the radiation pattern for either ground wave or combined ground wave and sky-wave radiation as desired. The Daventry station of the B.B.C. actually varies the angle of radiation when changing from one transmission to another;[2] but so far no theoretical investigation has been carried out successfully.

The author has pointed out in another paper* that among other factors, the angle of radiation plays an important part in determining the propagation condition of sky-waves. In this paper, the values of maximum usable angle of radiation have been mathematically calculated and graphically shown when the maximum usable frequencies, the virtual height of the layers and their electronic densities and the distances to be covered are known. Also the values of changes in the angle of radiation

* "Max. Usable Frequencies for Sky-wave Transmission" (still under publication).

corresponding to changes in the heights and densities of reflecting layers (in order to keep the same skip distance) have been calculated taking into consideration the curvature of the earth, the curvature and thickness of the ionosphere. The limits of this change have also been mathematically discussed and calculated for a certain critical frequency. Lastly, the effects of this change on the Field Strength of the reflected wave have been calculated for the case of vertical half wave and quarter wave antennas but a detailed discussion of this will form the subject of another paper.

The Angle of Radiation

The angle which the radiating aerial makes with the surface of the earth is as important as the maximum usable frequency in the sky wave transmission. The maximum possible distance for single reflection transmission corresponding to zero angle of departure is about 2400 K. M. for E layer transmission and 3500 to 4000 K. M. for F2 layer depending upon the height of the layer as well as the maximum electronic density which varies with height.

This ideal condition cannot be applied for practical radio communication on short waves as the absorption due to earth is very great, as it varies inversely as the square root of the frequency used. Hence $3\frac{1}{2}^\circ$ is assumed to be lowest limit of the angle of departure over the maximum distance of 1700 K. M. for E layer and 2800 to 3000 K. M. for F2 layer depending again on the virtual heights and electronic densities of the layers.

For the present, single hop transmission upto 3500 K.M. is considered which is nearly correct for higher distances as well, as this is the highest attainable range for single hop transmission. Greater distances are obtained by multiple reflections in which no increase in maximum usable frequency or maximum usable angle of radiation is possible.

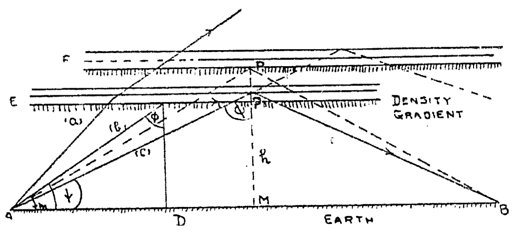


Fig. 1.

The effects of angle of incidence at the ionosphere and the angle of radiation at the earth are shown in the case of a reflection from the ionospheric layers in fig. (1). It is evident from fig. (1) that a wave striking the density gradient may be refracted until it is at an angle either equal to or greater than the critical angle.

The amount of attenuation in the ground waves is great while the amount of bending produced by the ionosphere becomes less. For frequencies above 6 Mc/s. the ground wave is attenuated below useful audibility at a moderate distance from the transmitter before the skywave is returned to earth. This produces a skip distance zone where no signals are heard. The greater the angle of radiation, the shorter this skip distance, ψ_m is the maximum value of this angle of radiation at which waves will be reflected and for higher angles this ray will penetrate the layer. The greater the angle of radiation, the higher is the frequency required to penetrate completely all of the layers.

There are two methods of keeping the skip distance constant when the heights and densities of the reflecting layer change from time to time. In case the heights and densities increase as is the case when the reflecting layer is changed from E to F layer, then to keep the skip distance constant, the frequency is increased. This skip distance can remain constant in a different way, *i.e.*, keeping the frequency as before but increasing the angle of radiation as shown in the fig. (1). For the E layer and reflection at point P, the angle of radiation is $\angle$PAB and for the F layer and reflection at point P′, the angle of radiation is $\angle$P′AB. The skip distance remains constant for the ray comes back to B again. The change in the angle of radiation so effected is $\angle$PAP′.

Suppose D is the skip distance.

R is the radius of curvature of earth.
H is the height of layer.
μ is the refractive index of the layer.
N is the maximum electronic density.

The skip distance can be calculated on the simple assumption that the wave instead of being refracted by the ionised layer undergoes a mirror like reflection at the point of maximum electronic density. The skip distance is given by the following equation.

$$\mu^2 = \frac{\sin^2 (D/2R)}{\sin^2 (D/2R) + \left[1 + \frac{h}{R} - \cos (D/2R)\right]^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Putting $\frac{D}{2R} = \theta$ where θ is the angular displacement around Earth to point of reflection.

Hence

$$\mu^2 = \frac{\sin^2\theta}{\sin^2\theta + \left[1 + \frac{h}{R} - \cos\theta\right]^2}.$$

In the case of flat ionosphere, the condition of reflection is given by $v=\sin\phi$ where ϕ is the angle of incidence at the point of reflection at the ionosphere as shown in the fig. (3).

Also
$$\sin\phi=\frac{\cos\psi}{1+\frac{h}{R}} \quad \ldots \quad (2)$$

$$\therefore \cos^2\psi=\frac{\left(1+\frac{h}{R}\right)^2\sin^2\theta}{\sin^2\theta+\left[1+\frac{h}{R}-\cos\theta\right]^2} \quad \ldots \quad (3)$$

$$=\frac{\left(1+\frac{h}{R}\right)^2\sin^2\theta}{1+\left(1+\frac{h}{R}\right)^2-2\cos\theta\left(1+\frac{h}{R}\right)} \quad \ldots \quad (4)$$

Suppose the value of D increases, so the value of $\cos^2\psi$ also increases, *i.e*, the more the skip distance the lower the angle of radiation.

DETERMINATION OF THE RELATION BETWEEN THE ANGLE OF RADIATION AND THE HEIGHT OF THE LAYER.

(*i*) *For the case of flat ionosphere and flat earth.*

Suppose $f_{\text{max.}}$ is the frequency of the wave at oblique incidence, *i.e*, $<\phi$ as shown in fig. (1).

Suppose fc is the critical frequency at normal incidence. By Secant's Law

$$f_{\text{max.}}=fc\sec\phi \quad \ldots \quad (5)$$

$$\frac{D}{2h}=\tan\phi \text{ where D is the skip distance}$$

Hence

$$\sec\phi=\sqrt{\frac{D^2+4h^2}{4h^2}} \quad \ldots \quad (6)$$

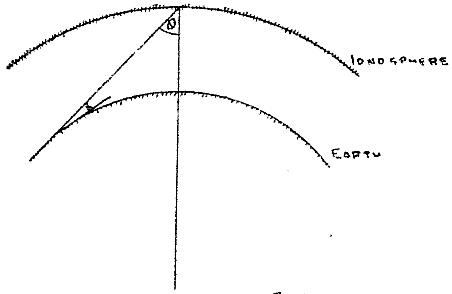
But
$$\sin\phi=\frac{\cos\psi}{1+\frac{h}{R}} \text{ from equation(2)}$$

$$=\cos\psi \text{ for } R\rightarrow\infty \text{ when earth is flat}$$

$$\therefore \sin\psi=\frac{2h}{\sqrt{D^2+4h^2}} \quad \ldots \quad (7)$$

The values of the angle of radiation for a distance of 500 K.M. only are correct as higher values of the distance involve the curvature of the earth which has been neglected in this calculation.

The real height of reflection is involved in all these calculations and the distinction between real height h and virtual height h' must be known for exact



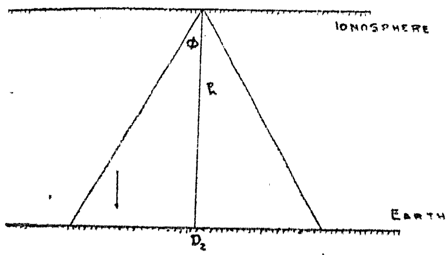


FIG. 2.

computation of propagation over a distance. In order to determine the angle of radiation (for different frequencies) exactly would require accurate knowledge of the difference between real and virtual heights at all frequencies and would also include the force and direction of Earth's Magnetic Field along the wave path in the ionosphere. The real and virtual heights have been assumed to be the same—an assumption which is quite correct for E region but less certain for the F regions, specially near the penetration frequencies. The range of wave frequencies close to penetration frequency leads only to an echo of secondary importance so that failure of the assumption here is not serious.

(ii) Case of flat Ionosphere and curved Earth.

With the same notations as before and considering the ionosphere to be flat, the value of ϕ_1 is the same as that of ϕ.

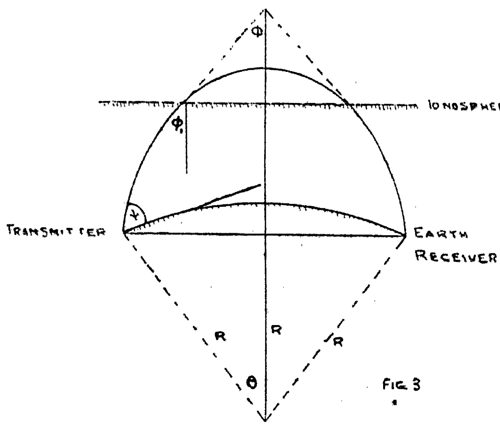


FIG. 3

$$\tan\phi = \frac{\sin D/2R}{1+\frac{h}{R}-\cos\ D/2R} \qquad (8)$$

Putting $D/2R=\theta$ as before

$$\tan\phi=\frac{\sin\theta}{1+\frac{h}{R}-\cos\theta}$$

From these two equations eliminating ϕ

$$\sin\psi=\sqrt{1-\frac{\left(1+\frac{h}{R}\right)^2\sin^2\theta}{1+\left(1+\frac{h}{R}\right)^2-2\cos\theta\left(1+\frac{h}{R}\right)}} \qquad (9)$$

(*iii*) *Case of thick and curved Ionosphere and curved Earth.*

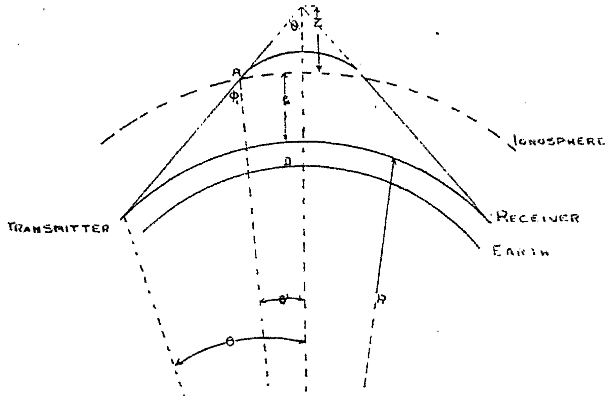


FIG4

Suppose,

ϕ_1 is the angle at the lower boundary of the ionosphere.

ϕ is the half vertex angle as shown in the fig. (5).

z is the height of the vertual reflecting point from the lower boundary of ionosphere. This includes the thickness of the ionosphere.

z_1 is the maximum height of the penetration and the distance of the highest penetrated point and the vertex.

h is the height of the lower boundary of the ionosphere.

In the case of thick ionosphere the angle made by the wave at the lower boundary of the ionosphere is not the same as the half of the vertex angle, but is given as below:—

By the Law of Sines

$$\text{Sin } \phi_1=\text{Sin } \phi\left(1+\frac{z}{R+h}\right) \qquad (10)$$

$$=\frac{\cos\phi}{1+\frac{h}{R}} \text{ from equation (?)}$$

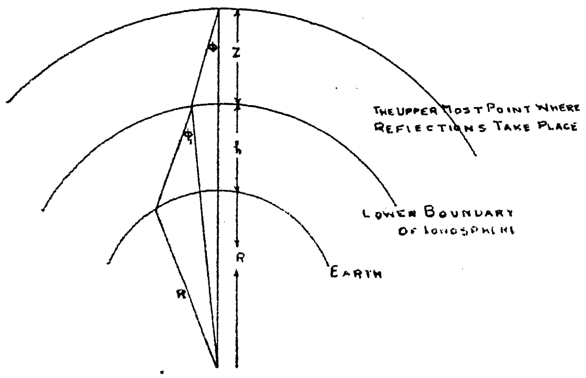


FIG 5

$$\tan \phi_1 = \frac{\sin D/2R}{1+\frac{h+z}{R}-\cos D/2R} \quad \text{from equation(8)}$$

Putting $D/2R=\theta$ as before

$$\tan \phi_1 = \frac{\sin \theta}{1+\frac{h+z}{R}-\cos\theta} \quad \ldots\ldots \quad (11)$$

$$\text{Hence } \sin \psi = \sqrt{1-\frac{\left(1+\frac{h}{R}\right)^2 \sin^2\theta}{1+\left(1+\frac{h+z}{R}\right)^2-2\cos\theta\left(1+\frac{h+z}{R}\right)}} \quad \ldots (12)$$

It will be noted that the values of the angles of radiation are less in the case when the thickness of the ionosphere is taken into consideration as compared with the case of flat ionosphere. So in this case either the maximum usable frequency should be increased or the angle of radiation decreased.

The effect of ionosphere is to alter the wave in the direction in which the resultant energy flows in such a manner as to cause the wave-path to bend away from the regions of high electronic density towards the regions of lower density. The magnitude of this effect varies with amplitude and average velocity of the electron vibrations and hence becomes increasingly great as the wave frequency is lowered.

The path that a radio wave follows in the ionosphere depends upon the way in which the refractive index varies with height above the earth. The bending of the ray is proportional to ionisation gradient or change of refractive index with height and is greatest when the electron density changes rapidly with height. Actual calculation of the wave paths cannot be carried out with any degree of

accuracy because of the lack of knowledge concerning the distribution of electron density with height in the ionosphere. But the condition of reflection can be

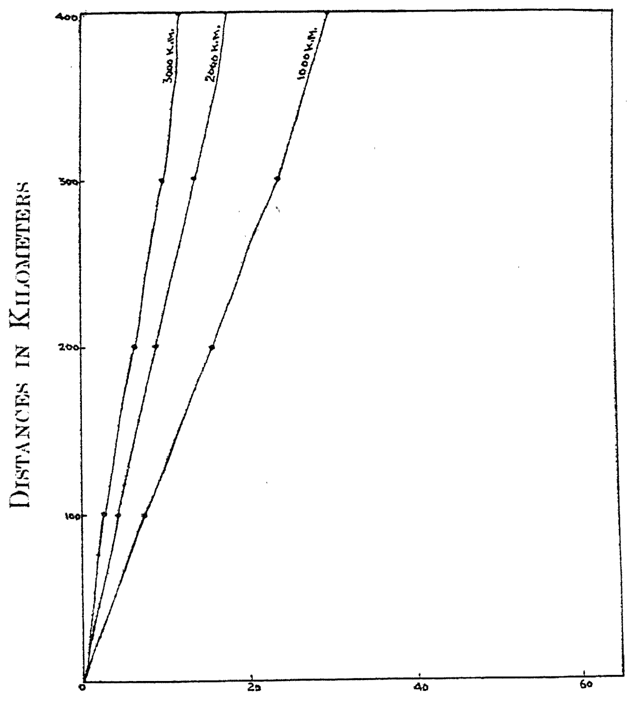


Fig. 6. Graphs showing the values of angles of radiation for various distances of transmission for the curved and thick parabolic ionosphere and curved earth.

calculated which will give the maximum and minimum values of angle of radiation as it is not possible to increase or decrease this angle beyond a certain limit.

The conditions of reflections for the flat and curved ionosphere are different due to the variation in the angle of incidence for the two cases.

(*i*) *Case of flat ionosphere.*

In the case of flat ionosphere, the condition of reflection is given by $\mu=\sin\phi$ where ϕ is the angle of incidence

But $$\mu=\sqrt{1-\frac{Ne^2}{\pi mf^2}}$$

$\therefore$ $$\sin\phi=\sqrt{1-\frac{Ne^2}{\pi mf^2}} \qquad (13)$$

where N=the max. electronic density of the layer.

e=the electronic charge.

m=mass of the ion.

f=the frequency in kc/s.

Also $$\sin\phi=\frac{\cos\psi}{1+\frac{h}{R}} \text{ from equation (2)}$$

Hence $$\sin\psi=\sqrt{1-\left(1+\frac{h}{R}\right)^2\left(1-\frac{81N}{f^2}\right)} \quad \ldots \ldots \quad (14)$$

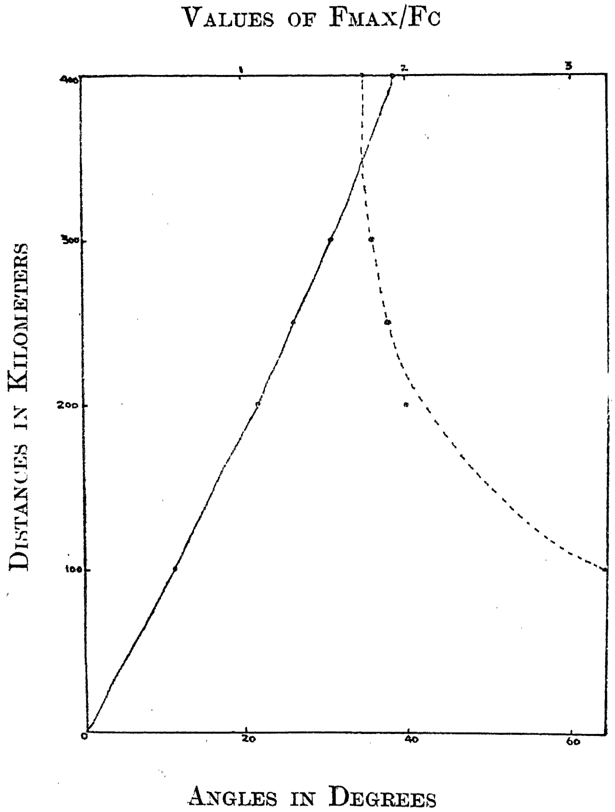
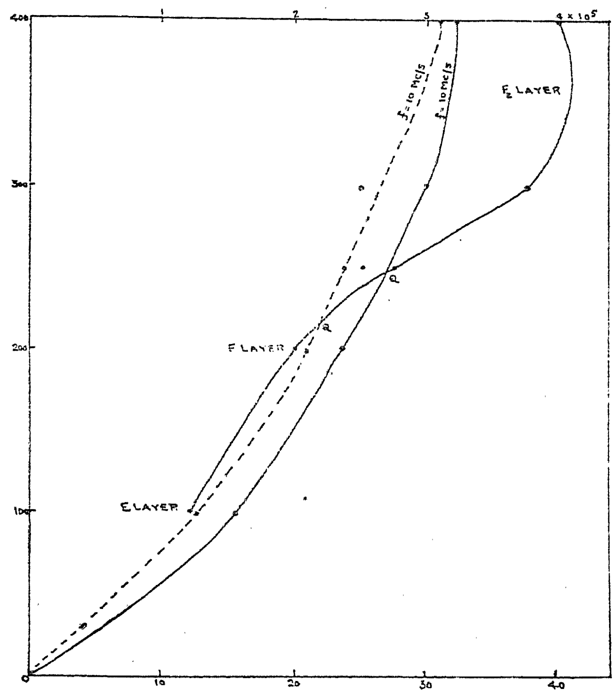


Fig. 7. Graphs showing the relation between various heights and angles of radiation and heights and values of multiplying factor $\frac{f\,max}{fc}$. The former is shown as —— and the latter as......

The values of $\sin\psi$ will be determined by the equation (14). Corresponding to any height h, there can be found the value of N, so the maximum value of $\sin\psi$

will be given by the above equation as $\mu = \sin\phi$ is the case of critical angle refraction. The minimum values of $\sin\psi$ can be any value above sin $3\frac{1}{2}°$. The minimum value of ψ at which sufficient energy is radiated to produce a readable signal varies with the terrain, the power of the transmitter and the sensitivity of the receiver. Over sea water ψ can be very nearly zero but over land the minimum value may be several degrees ; $3\frac{1}{2}°$ degrees being a fair average approximation.

Electronic Density



Distances in Kilometers

Angle in Degrees

Fig. 8. Graphs showing the values of angles of radiation for the case of flat and curved ionosphere. Superimposed on these is the typical ionospheric trace which determines the propagation condition for different layers for a frequency of 10 MC/S. The curve for the flat ionosphere is shown as—— and the curve for the curved and thick ionosphere is shown as .....

The maximum values of $\sin\psi$ are calculated for the E, F1, F, F2 layers by taking their normal values of heights and maximum electronic density. The value of $f = 10$ Mc/s in each case.

These electronic densities have been calculated on the assumption that Lorentz Polarization—correction[4]—factor is 1/3 instead of zero as used to be supposed so

far. Ultimate decision in this matter must await discussion of the experimental evidence in the literature, but it now appears quite conclusive that in calculating N Lorentz factor must be assumed to be 1/3 for high frequencies at least.

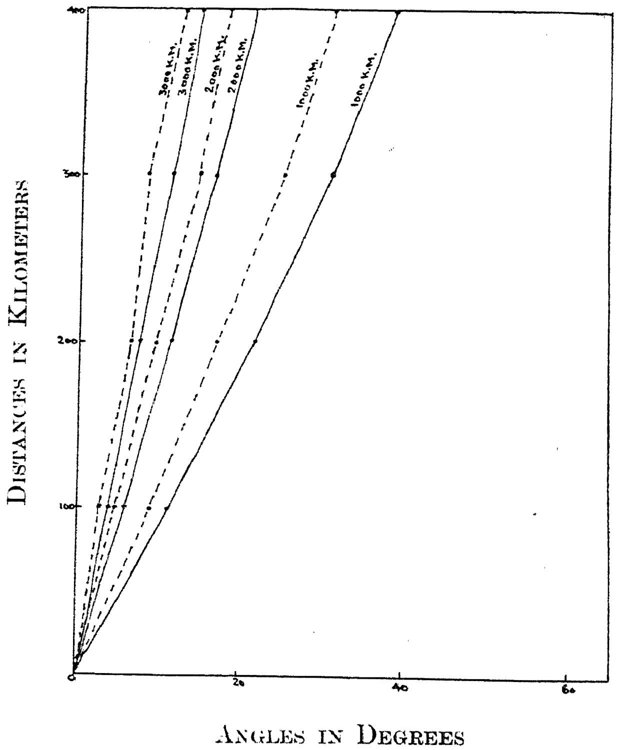


Fig. 9. Graphs showing the values of angle of radiation for various distances of transmission. The numbers on the graphs refer to distances in kilometers. The graphs for the flat ionosphere and flat earth are shown as—— The graphs for curved ionosphere and curved earth as......

(*ii*) *Case of Curved and Thick ionosphere.*

The wave will penetrate the curved ionosphere until the refractive index μ is reduced to the value.

$$\mu=\sin\phi \qquad \text{as before.}$$

As already shown, for the case of curved and thick ionosphere $\phi_1 \neq \phi$, where ϕ_1, is the angle of incidence of the wave on the lower boundary of the ionosphere, but is given by the following relation :—

$$\sin\phi=\frac{\sin\phi_1}{1+\frac{Z_0}{R+h}} \qquad (15)$$

Where Z_0 is the maximum height of penetration above this lower boundary.

Hence $$\sin\psi=\sqrt{1-\left(1+\frac{h}{R}\right)^2\left(1+\frac{Z_0}{R+h}\right)^2\left\{1-\frac{81N}{f^2}\right\}}$$

This gives the maximum value of $\sin\psi$. Under practical conditions Z_0 is almost always less than 400 kilometers. So in this calculation of maximum $\sin\psi$ determination, 400 K.M. can be taken to be the value of Z_0. Again corresponding to any value of h, the value of N can be found out by E, Chapman's[3] graphs, which have been further developed by Millington[5]. The minimum value of $\sin\psi$ is the same as in the previous section.

The maximum value of sin for E, F1, F, and F2 layers are calculated by taking the values of heights and electronic densities.

It will be observed that the values of maximum $\sin\phi$ are less than those obtained for the case of flat ionosphere. *i.e.*, for the curved ionosphere. lower values of $\sin\psi$ are to be made use of.

VARIATION OF MAXIMUM ELECTRONIC DENSITY WITH HEIGHT OF THE LAYER

The greater part of the work carried out during the last few years has been concerned with the determination of maximum electronic density for different ionospheric regions in different parts of the world. Chapman has dealt with this problem mathematically. He has examined the ionising effect of monochromatic Solar radiation in an atmosphere on rotating earth. Millington[5] has further elucidated Chapman's work by preparing a number of charts indicating the theoretical variations of ionisation in a form easily comparable with experimental results.

Suppose that molecular density P of an ideal atmosphere varies with height h according to the formula

$$P=P_0 \frac{-h}{e\,H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where H is constant and $=8{\cdot}4$ K. M.

Suppose I is the rate of absorption of radiation by the atmosphere at height h when incident at angle X. Then it may be shown that

$$I=I_0 e^{1-z-e-z\sec X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\text{Where } Z=\frac{h-h_0}{H}.\ I_0=\frac{S\infty}{H\ expl} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

A is a constant, being the absorption coefficient, $S\infty$ is the intensity of radiation outside the earth's atmosphere ($h=\infty$). Thus Z is the height measured in terms of H as unit reckoned from the level h_0.

In the wireless experiments, we are concerned not with the rate or ionisation (q) but with N the ionic density. These quantities are related by the equation

$$\frac{dN}{dt}=q-\alpha N^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where α is the coefficient of recombinations and t the time. Chapman's results are plotted in a series of curves illustrating the variation of N with time and with height. These give the values of maximum electronic density corresponding to any height and time.

As the region F2 behaves abnormally for this the variation of N is given as below:—

$$\frac{dN}{dt}=q-\frac{N}{T}\frac{dT}{dt}-\alpha N^2 \quad \ldots \ldots \quad (21)$$

where T is the absolute temperature of atmosphere.

The first term in this equation expresses the rate of ion production by the solar radiation and the second term expresses the rate of concentration of electrons, ions and molecules resulting from thermal expansion or concentration due to solar heating.

From these graphs and equations the value of maximum electronic density corresponding to any height is determined and consequently the values of maximum sin ψ are calculated as suggested before.

Effect of the change in angle of Radiation upon the Field Strength.

Most of the absorption suffered by wave passing through the ionosphere takes place at the lower edge of the ionosphere where the atmospheric pressure is greatest and absorption taking place high up in the layer is relatively small. The absorption will also be less the higher the frequency because the average velocity of the vibrating electrons decreases as the frequency is increased[6].

The Sky-wave in travelling between the earth and the ionosphere suffers attenuation due to spreading of the waves in such a manner that the field strength is inversely proportional to distance. If F1 or F2 is the reflecting layer the loss due to E layer is inversely proportional to square of the frequency used and becomes smaller the more nearly vertical the angle of radiation.

Suppose,

Q is the loss of energy in the ionosphere.

F is the frequency used.

ψ is the angle of radiation.

$$Q=K\left[f_1\left(\frac{1}{F^2}\right)+f_2\left(\frac{1}{\psi}\right)\right] \quad \ldots \ldots \quad (22)$$

If E is the reflecting layer the attenuation in this region depends upon the way in which the electrons are distributed with height and becomes greater as the angle of radiation approaches vertical. For a single hop transmission the ground losses do not come in but for multiple reflections the ground attenuation is more than the loss in the ionosphere,

Hence in the case of change of height of the reflecting layer or change of the reflecting layer from E to F1 or F2, if the angle of radiation is increased instead of

decreasing or increasing the frequency (as is usually done) the new field strength will be greater because the loss in the ionosphere is reduced.

Also the value of field strength is as given below[7] :

$$E=\frac{Ah_s\ I_s}{\lambda\sqrt{R\sin\theta}}\,e^{\frac{-\alpha D}{\lambda^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where R is the radius of earth, θ is the angle subtended by the stations at the centre of the earth, h_s is the effective height of the transmitting aerial, I_s is its aerial current, λ is the wave-length.

α is given as

$$\alpha=\frac{1}{2\sqrt{2ch}}\left[\left(\frac{\mu_1}{\phi'_1}\right)^{\frac{1}{2}}+\left(\frac{\mu_2}{\phi_2}\right)^{\frac{1}{2}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where c is the velocity of electromagnetic waves, h is the height of the ionosphere, μ_1 & μ_2 the permeabilities of the layer and earth which can be taken to be unity ϕ'_1 is the conductivity of the ionised layer, ϕ_2 is the conductivity of the earth, A is a constant.

The angle of radiation is proportional to h/λ in the case of vertical half wave and quarter wave aerial. Hence the value of E increases as the angle of radiation is increased and the loss in the ionosphere is reduced.

In the case when the heights and densities of the layers decrease, the angle of radiation will have to be lowered to maintain the same maximum usable frequency for getting the same skip distance. The aerial system will require some alterations in this case in order that Field Strength and overall merit of the transmitter may remain constant.

Conclusion

The mathematical calculations aud graphs can be utilised in studying the corelation of high frequency sky-wave transmission with regular ionospheric observations. With the help of extensive knowledge of maximum electronic density and heights for the different times during the day and night, their diurnal and seasonal changes as well as their changes with solar sun spot cycle, it is possible to adjust these changes by changing the angle of radiation instead of changing the frequencies from time to time, or by changing both. The skip distance will remain the same and the field strength will increase and also the attenuation will be less with the increment in the angle of radiation as shown before.

Suppose for a range of 1000 K. M. at 17·00 the maximum usable frequency is 12·32 Mc/s and the reflecting layer is F2. At 21·00 the reflecting layer changes to F and the maximum usable frequency becomes 5·28 Mc/s. Hence the corresponding change in frequency is 7·04 Mc/s. From the fig. (6) the corresponding change is given by ψ and is equal to $7°35'$.

If the graph in fig. 10 is superimposed on a typical ionospheric trace, its intersection gives the layer from which a certain frequency at a certain angle of

radiation will be reflected to a certain distance. As for example in the graph fig. 10 the curve for a distance of 1000 K. M. and frequency 10 Mc/s intersects the layer at point Q and corresponding to this point of intersection, the angle of radiation is 27°. This shows that if a frequency of 10 Mc/s is to be reflected from F layer to a distance of 1000 K. M. the angle of radiation should be 27°. For higher values of angle of radiation, no reflection will take place but for lower values the skip distance will increase. The value for the curved ionosphere will be 21°30′.

When the angle of radiation will be changed, the ground waves will be reduced in intensity and the region of fading will be pushed forward. The net result will be that the skywave will be negligibly small until the strength of the ground waves becomes too weak and will then come in with a strength somewhat greater than that of ground wave. The sky-wave will not reduce the night coverage and at the same time will keep the region of high distortion to a minimum.

Another application of this can be made in the design of transmitting aerials. Suppose from ionospheric data of the region to be served it is found that the height of the reflecting layer changes from h_1 to h_2 for a frequency f_1. The corresponding change in the angle of radiation is readily found from the graphs or calculations and also its optimum value can be determined so that with a new frequency $f_2>f_1$ the aerial gives a fairly satisfactory signal strength throughout the whole transmission.

It is hoped if an aerial system is set up in which the angle of radiation can be varied, these results can be experimentally verified and on the basis of theory out-lined above exceptions in most cases can be accounted for.

ACKNOWLEDGEMENT

This investigation was undertaken at the suggestion of Dr. Wali Mohammad to whom the author expresses his sincere thanks for constant help and guidance. The author also thanks Dr. K. N. Mathur for his valuable help and suggestions and acknowledges with thanks the reproduction of figs. (4) and (5) from N. Smith's[8] paper.

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A CONTRIBUTION TO THE EMBRYOLOGY OF THE GENUS *POLYGALA*

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Communicated by Dr. P. Maheshwari

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SUMMARY

1. The development of the anther proceeds normally. The parietal tissue consists of an endothecium, two middle layers and tapetum. The tapetal cells occasionally become binucleate.

2. Quadripartition of the microspore mother cells takes place by centripetal furrows. At the shedding stage the pollen grains contain a large tube nucleus and a small generative cell.

3. During the development of the ovule, the hypodermal archesporial cell divides to form the megaspore mother cell and a parietal cell. The chalazal megaspore of the linear tetrad develops into the normal eight-nucleate embryo sac.

4. In *P. arillata* the antipodals are ephemeral and soon disappear; in *P. chinensis* their number increases to four, each becoming binucleate; in *P. sibirica* there are several uninucleate antipodal cells.

5. After fertilization the lower end of the embryo sac elongates to form a tubular structure, here designated as the "Endosperm process" in *P. chinensis* and *P. sibirica*. This has a haustorial function. It is absent in *P. arillata*.

6. A few stages of embryo development are also described.

Introduction

A perusal of the existing literature on the Polygalaceæ shows that our knowledge of this family is rather meagre. The chief contributions are by Cardiff (1906), Wirz (1910), Shadowsky (1912) and Jauch (1918). These have already been summarised by Schnarf (1931) and Schürhoff (1926) and it is unnecessary to cover the same ground again. The latest paper is by Karsmark (1933) on *Polygala comosa*. The anthers are monothecous with 2 loculi and the maturation divisions of the pollen mother cells are simultaneous. The ovule has a small "arillus" or caruncula. The development of the embryo sac proceeds normally but the antipodal cells are large and conspicuous. The endosperm is free nuclear and forms a micropylar as well as chalazal accumulation of plasm and nuclei. At the chalazal end there is a narrow pouch.

The present paper deals with 3 species of the genus *Polygala*—*P. chinensis*, Linn, *P. arillata*, Ham., and *P. sibirica*, Linn.

F. 11

Material and Method

Flowers and fruits of *P. arillata* and *P. sibirica* were collected from the hills of Kodaikanal in the month of October. *P. chinensis* was collected at Bangalore in November and December. The materials were fixed in Bouin's fluid between 9 A.M. and 3 P.M. on bright days. Sections were cut at 10μ for the study of microsporogenesis, but thicker sections were found more suitable for the embryo sac and embryo. Heidenhain's iron-alum hæmatoxylin was generally used throughout the study.

Organogeny

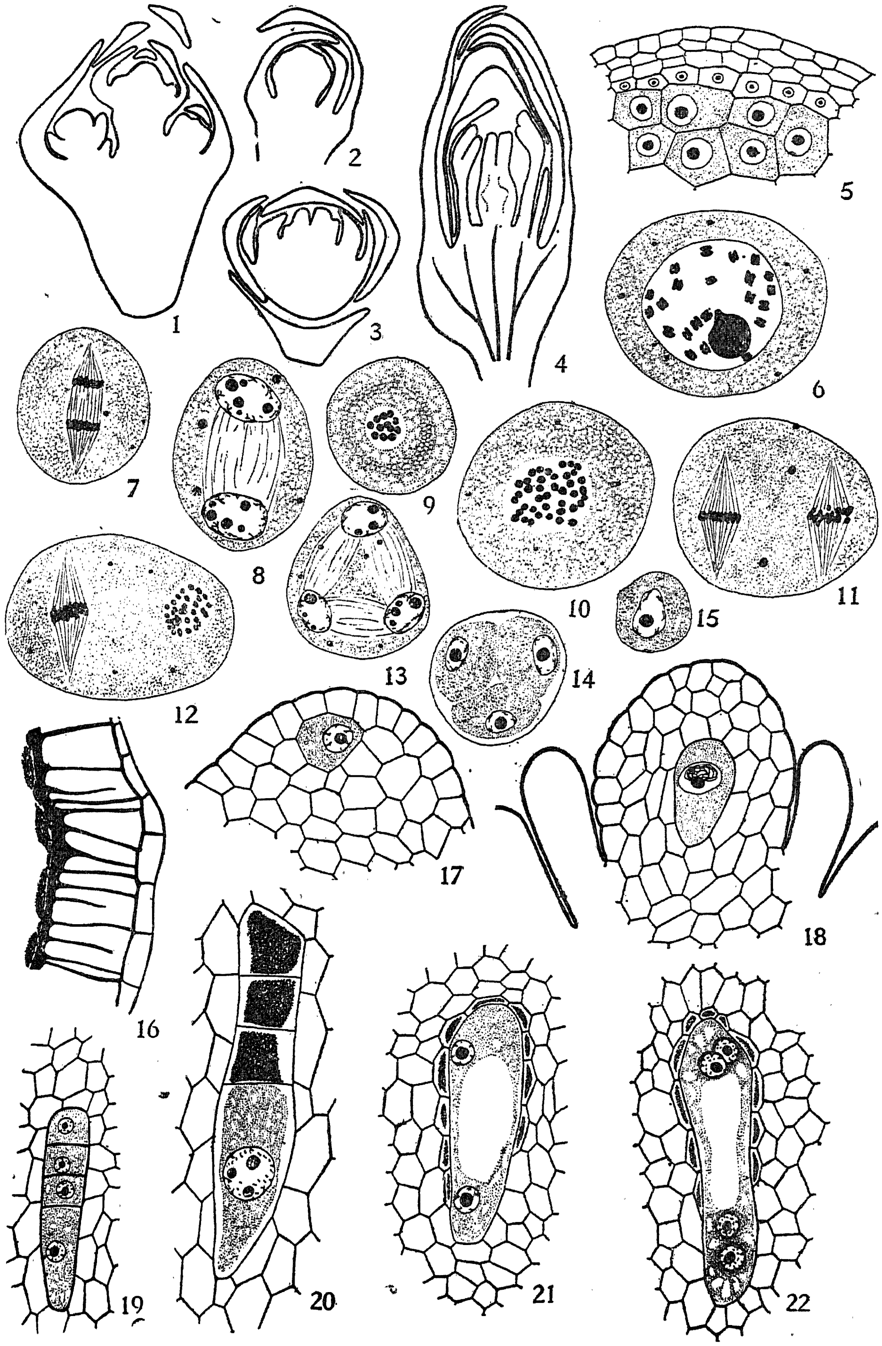
A reference to figure 1—4 shows that the organogeny is perfectly normal. The first whorl to develop is that of the sepals (Fig. 1). The calyx lobes bend over and cover the floral cone before the other parts are differentiated. The petals arise next (Fig. 2), then the stamens (Fig. 3) and last of all the carpels (Fig. 4).

Development of the Anther and Male Gametophyte

The hypodermal archesporial tissue is conspicuous even from the earliest stages by the large size, rich cytoplasmic contents and prominent nuclei of its cells. The archesporial cells divide periclinally into an outer and an inner layer of cells. The former gives rise to four parietal layers, the outermost and the innermost of which constitute the endothecium and the tapetum respectively, with the two middle layers in between. The inner layer of cells undergoes repeated divisions to form a number of microspore mother cells (Fig. 5). The tapetal layer is very conspicuous from the earliest stages. Some of the cells are very large while others are comparatively small. Mostly they are uninucleate but some may occasionally become binucleate. They begin to disorganize soon after the reduction divisions are over. During meiosis certain darkly staining extra-nuclear bodies are found in the cytoplasm of the microspore mother cells and they persist until the separation of the microspores from the tetrad (Figs. 6 - 8 and 10—15). During diakinesis, the bivalents are distributed throughout the nuclear cavity (Fig. 6). Usually the nucleolus persists for a long time. A polar view of the metaphae shows the haploid number of chromosomes to be 12 in *P. chinensis* (Fig. 9) and 40 in *P. arillata* (Fig. 10). The spindles of the second division are at right angles or parallel to each other (Figs. 11 and 12). At the completion of the divisions, the four daughter nuclei are separated by quadripartition furrows proceeding from the periphery inwards (Fig. 14). The spores lie free in the cavity of the original mother cell and come out when the wall disorganizes. The pollen grains are spherical with a thick and warty exine and a thin intine. At the shedding stage they have one large tube nucleus and a small

PLATE I

D. SRINIVASACHAR—*A Contribution to the Embryology of the Genus Polygala.*



generative cell. The mature anther shows the degenerated remains of the middle layers and tapetum, an endothecium of elongated cells with fibrous thickenings and the epidermis (Fig. 16).

DEVELOPMENT OF THE OVULE AND EMBRYO SAC

In all the three species, the nucellar primordia begin to be visible only when the microspore mother cells are in the prophase of meiosis. The hypodermal archesporial cell is distinguishable by its rich cytoplasmic contents and prominent nucleus (Fig. 17). It divides to form an outer parietal cell and an inner sporogenous cell. The former, by a series of periclinal divisions, forms a number of parietal layers which bury the mother cell deep into the nucellus (Fig. 18). The mother cell enlarges in size and its nucleus passes through the usual divisions and forms a linear tetrad of megaspores (Fig. 19). In all the three species, the chalazal megaspore alone survives, while the remaining three degenerate (Fig. 20). This is also the case in *Polygala comosa* investigated by Karsmark (1933).

The functioning megaspore enlarges rapidly and becomes vacuolate. The nucleus undergoes three successive divisions and forms a normal eight-nucleate embryo sac (Figs. 20—25). This continues to enlarge at the expense of the surrounding nucellar cells, remnants of which persist as darkly staining masses all round the embryo sac.

The fully organized embryo sac has two synergids and an egg at the micropylar end, two polar nuclei in the centre and three antipodal cells at the chalazal end. The micropylar end is broad, while the chalazal end is narrow. In *P. arillata* the synergids are comparatively small and provided with basal vacuoles and prominent beaks (Fig. 23). In *P. sibirica* they are large with a well-developed filiform apparatus and a hyaline area at the tip (Fig. 26). In *P. chinensis* they are large with basal vacuoles (Fig. 24). The polar nuclei lie side by side for a long time and fuse only on the arrival of the second male nucleus. This fusion takes place at the centre of the embryo sac in *P. arillata* and *P. chinensis* and slightly lower down in *P. sibirica.*

To start with, there are three distinct antipodal cells in all the species. In *P. arillata,* they already show signs of disorganization at the time of fertilization and are completely disorganized by the time the fertilized egg divides. In *P. chinensis,* one of the antipodal cells divides into two and the nucleus of each of these four cells divides once, so that each cell becomes binucleate (Fig. 27). These cells persist for a long time even after fertilization and seem to supply nutrition to the growing embryo until the endosperm takes over this function. In *P. sibirica,* a further step is seen in the development of the antipodal. All the antipodals may

divide followed by wall formation, thus giving rise to a mass of 7-12 cells (Fig. 28). They persist for a long time and their remnants are seen at the tip of the "Endosperm process" which is to be described shortly. The nucellar cells situated below the antipodal end of the embryo sac are comparatively large and take a deep stain and act as nutritive cells. The outer and the inner epidermal layers of the outer integument consist of long palisade-like cells, particularly at the micropylar end of the embryo sac (Fig. 29).

Fertilization

A number of pollen tubes are seen in the stylar canal. Only one tube enters an embryo sac and it destroys either one or occasionally both the synergids (Figs. 30 and 31). One of the two male nuclei approaches the egg nucleus and the nuclear membrane breaks down at the point of contact. Both the male and female nuclei are in the resting condition at the time of fusion. The second male nucleus passes down the embryo sac and meets the two polars to form a large endosperm nucleus (Figs. 32 and 33).

Endosperm Formation

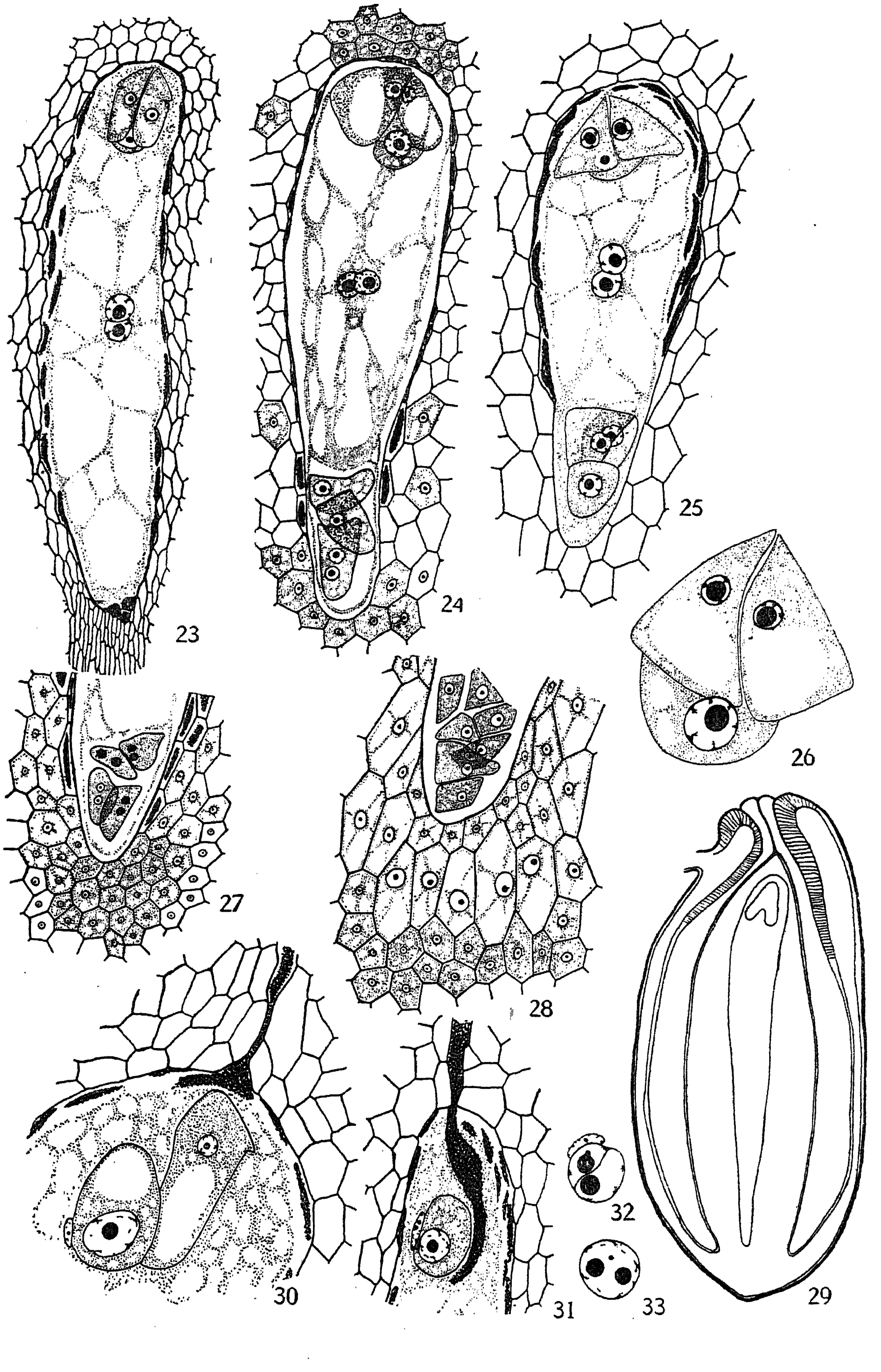
The endosperm is free nuclear and the resulting nuclei possess two or more nucleoli. After a large number of free nuclei are formed, wall formation commences at the micropylar end and gradually proceeds downwards. In all cases, the endosperm nuclei at the antipodal end remain free throughout. The lower region of the embryo sac which penetrates into the nucellus becomes a haustorial organ and is referred to in this paper as the "Endosperm process" (Figs. 34—36).

Development of the Embryo

The egg divides immediately after fertilization. Its nucleus often contains a large and a small nucleous, the latter being evidently derived from the male nucleus (Fig. 37). By a transverse division the fertilized egg becomes two-celled. The distal cell divides once transversely forming a proembryo of three cells (Fig. 38). The apical cell of the proembryo divides by a vertical wall, followed by a transverse one, thus forming a quadrant of cells (Figs. 39—42). This undergoes further divisions to give rise to a large spherical mass of cells (Figs. 44—46), in which the outermost layer soon differentiates as the dermatogen (Fig. 45). Meanwhile, the second cell of the three-celled proembryo undergoes a vertical division (Figs. 40, 43 and 44), and after some further divisions it contributes to the embryo on the lower side. In *P. arillata* and *P. sibirica,* the basal cell also divides by a vertical wall so that in the mature embryo the suspensor consists of two cells placed side by side (Fig. 46). In *P. chinensis*, the basal cell by two or three transverse divisions forms a row of cells and in each a vertical division takes place so that the mature embryo has a suspensor composed of several tiers of cells.

PLATE II

D. SRINIVASACHAR—*A Contribution to the Embryology of the Genus Polygala.*



Discussion

The increase of the antipodal tissue in *P. chinensis* and *P. sibirica* probably aids in the nutrition of the embryo sac until the endosperm takes over this function. In *P. sibirica* it persists even in late stages when embryo formation is far advanced. Further, the lower end of the embryo sac penetrates into the nucellus carrying the antipodals along with it, so that even at a late stage their degenerated remains can be seen at its tip. In this "Endosperm process" the darkly staining cytoplasm has a large number of free nuclei which in later stages break up into small bits and are ultimately absorbed by the growing embryo. The formation of this haustorial process, the presence of multiple and persistent antipodals and the occurrence of deeply staining cells at the chalaza, show that in *P. sibirica* and *P. chinensis* there is a well-developed mechanism for the nutrition of the growing embryo.

Acknowledgments

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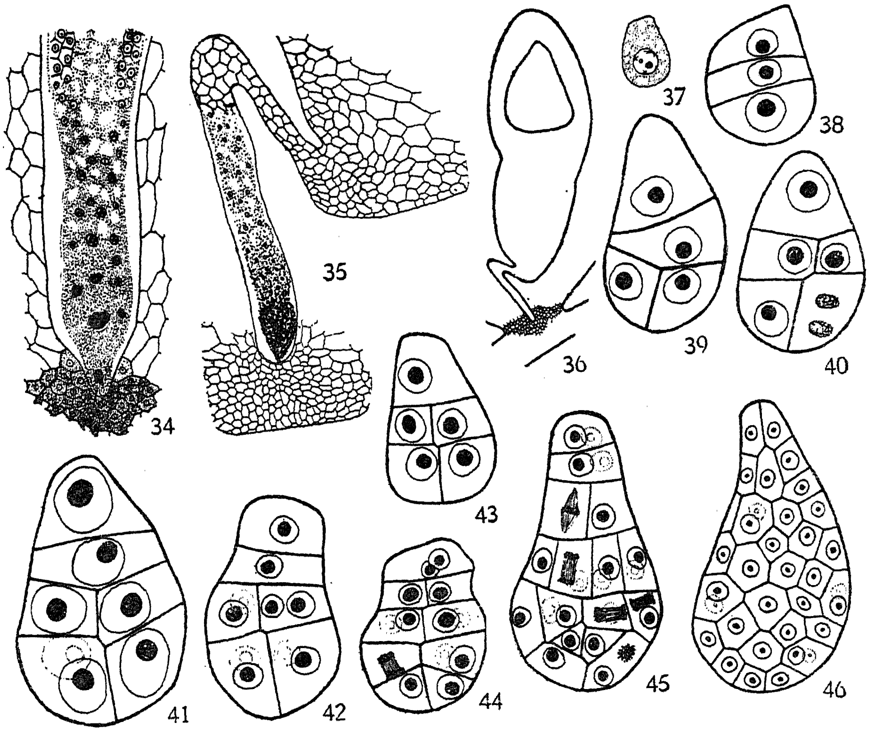
Explanation of Figures

Figures	*Description*
1 to 4	Stages showing sequence of development of the floral parts of *P. arillata*. Figs. 1—3, X80, Fig. 4, X50.
5	Part of a young anther of *P. arillata* showing the sporogenous layer, the tapetum, two wall layers and the epidermis. X800.

6 to 8 Stages in the development of the microspores of *P. arillata.*
10 to 14 Note the black extranuclear bodies. Figs 6—8, 10 and 11, X2700. Figs. 12—14, X1800.
9 Metaphase plate of first reduction division in *P. chinensis.* X3600.
10 Metaphase plate of first reduction division in *P. arillata.* X2700.
15 Microspore of *P. arillata.* X1800.
16 Part of an old anther of *P. arillata* showing the epidermis, the endothecium and a few degenerated cells of the middle layers and tapetum. X800.
17 Nucellus showing the hypodermal archesporial cell in *P. sibirica,* X1800.
18 Megaspore mother cell of *P. chinensis.* X. 1800.
19 Tetrad of megaspores of *P. chinensis.* X1800.
20 Tetrad of megaspores of *P. arillata,* showing degeneration of the upper three spores. X1800.
21 to 22 Two—and four nucleate embryo sacs of *P. sibirica* and *P. chinensis.* X1800.
23 to 25 Fully formed embryo sacs of *P. arillata, P. chinensis* and *P. sibirica* respectively. Fig. 23, X560; Figs. 24 and 25, X1800.
26 Egg apparatus of *P. sibirica.* Note the filiform apparatus in the synergids. X2700.
27 Four binucleate antipodal cells of *P. chinensis.* X1800.
28 Multiple antipodals of *P. sibirica.* Observe the rich cytoplasmic contents of the cells at the antipodal region of the embryo sac in this and the preceding figure. X1800.
29 A diagram of an ovule showing the elongated cells of the outer integument in *P. sibirica.* X80.
30 to 31 Fertilization in *P. chinensis* and *P. arillata.* Fig. 30, X1800. Fig. 31, X800.
32 to 33 Stages in triple fusion of *P. arillata* and *P. chinensis.* Fig. 32, X800. Fig. 33, X1800.
34 to 35 The "Endosperm process" of *P. sibirica,* and *P. chinensis.* X800.
36 An outline drawing of the ovule of *P. chinensis* showing the position of the "Endosperm process." X160.
37 Fertilization of the egg of *P. arillata.* X800.
38 to 46 Stages in the development of the embryo. Figs. 39 and 40. *P. sibirica,* X1800; Figs. 38 and 42—45, *P. chinensis,* X1800. Figs. 41 and 46, *P. arillata.* Fig. 41, X1800. Fig. 46, X800.

D. SRINIVASACHAR—*A Contribution to the Embryology of the Genus Polygala*



PROTEID YOLK FORMATION IN THE OOCYTES OF OPHIOCEPHALUS PUNCTATUS

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(Received on December 18, 1941)

The problem of vitellogenesis is one of the most intricate ones of cytology. Whereas it is possible that much of the discrepancy in the reports presented by various authors as to the mode of yolk formation may be due to a real diversity in vitellogenesis in the eggs of different animals, the conflicting accounts based on the investigation of the same material must be ascribed to misobservation and misinterpretation of facts on the part of the investigators. One remarkable example of this bewildering discrepancy is furnished by the writings of Nath (1931) and Narain (1930) in which these authors set forth the results obtained by a study of the oocytes of Ophiocephalus. According to Narain (1930) "the individual Golgi grains become lodged inside vacuoles and begin to swell up till they get converted into fairly big spheres." Again, "the secretion of yolk by the mitochondria starts at a time when a good deal of Golgi yolk has been already produced. The granular mitochondria begin to swell up till they form round discs of albuminous material." In a later paper (1937) Narain again discusses the mode of proteid yolk formation. He writes, "I have been able to trace only two types in fishes—the albuminous or mitochondrial yolk and the fatty or the Golgi yolk. The albuminous yolk is formed by the gradual swelling of mitochondria to form fairly big yolk discs, seen beautifully by Régaud Tupa's and Champy's methods." Further on he writes, "I am unable to confirm the view of Nath (24) that in Ophiocephalus the albuminous yolk is formed by the neutral red staining vacuoles and not by mitochondria." Hibbard and Parat (1928), on the other hand, come to an entirely different conclusion on the basis of the evidence furnished by the study of the oogenesis of Perca fluviatilis L. and Pygosteus pungitius L. They do not mention the presence of "fatty yolk" in the eggs of these fishes, but in an advanced oocyte of Pygosteus they describe "des gouttes d'huile(h) souvent volumineuses." It is obvious that what Narain (1930 and 1937) and Nath (1931) consider to be fatty yolk bodies Parat and his collaborator consider to be free fat droplets, in no way related to any cytoplasmic component. About the mode of origin of proteid yolk the latter authors write "Mais le chondriome forme surtout une sorte de feutrage autour des groupes vacuolaires périphériques; ces groupes sont constitués

d'assez nombreuses petites vacuoles qui prennent le Rouge Neutre et s' imprègnent parfaitement à l'argent et à l'osmium. Puis cette colorabilité et cette aptitude à l 'imprégnation diminuent progressivement au fur et à mesure de la concentration du contenu vacuolaire, concentration qui aboutit à la formation du vitellus. Ce dernier augmente de volume, le vacuome disparaît, et avec lui les dictyosomes, de l 'appareil de Golgi." Similar conclusions, it may be mentioned in passing, have also been arrived at by Hibbard in Discoglossus (1928), Marguerite Parat in a bear (1927) and Maurice Parat in certain Molluscs and Coelentrates (1927), Parat (1927) writes, "L'ovocyte, de la Méduse Chrysaora hysocella (L) est riche en vacuoles colorables par les colorants vitaux : il en est de même pour l'ovocyte de certains Mollusques Nudibranches come Aplysia punctata (Cuv) et Polycera quadrilineata, ou celui de l'Oursin Strongylocentrotus lividus (Lk). Le contenu des vacuoles semble au cours de l'ovogénèse se condenser pour aboutir à la formation de jeunes plaquettes vitellines, surtout protéiques et ne renferment encore point de lipoïdes."

Although Nath and Nangia (1931) do not accept Parat's vacuome hypothesis according to which the classical Golgi apparatus is represented in the normal living cells as a system of vacuoles capable of staining with neutral red, their own account of proteid yolk formation agrees in all essential details with that of Parat and his school. Nath and Nangia write, "When a young oocyte (Fig. 1) of Ophiocephalus is studied in a drop of normal saline, the most prominent inclusions are certain whitish droplets distributed at random in the cytoplasm. These have been identified as the vacuome of Parat and Hibbard. They are stainable with neutral red . . . . sometimes the stained contents of a droplet may shrink and appear as a red grain lying in the centre of a clear circular area . . . . ," and again "In the earliest oocytes the vacuoles are small and comparatively delicate. In the course of oogenesis they grow in size and their contents become denser as the result of the deposition of albuminous material inside their interior. Consequently, when an oocyte measuring about ·2 mm. (Fig. 17) is ruptured, the vacuoles which are now packed with albuminous material show a firm consistency . . . . " "Ultimately the vacuoles form the albuminous yolk sensu strictu, as has been very rightly claimed by Hibbard and Parat for Perca and Pygosteus and by Hibbard for Discoglossus."

The investigation conducted by the present writer on the ovarian eggs of Ophiocephalus does partially bear out the account of proteid yolk formation as given out by Nath and Parat and their collaborators. At a certain stage of growth, the cytoplasm of the egg does show some vacuoles each containing a sharply staining granule (Fig. 1). At first this granule is quite small in size, but gradually it grows bigger and ultimately comes to occupy the entire vacuole. It can now be recognized as a fully formed albuminous yolk spherule. Its staining reactions and resistance

to the action of acid in Bouin's fluid leaves no doubt that it is proteid in nature. These vacuoles and the granules contained in them are stainable with neutral red

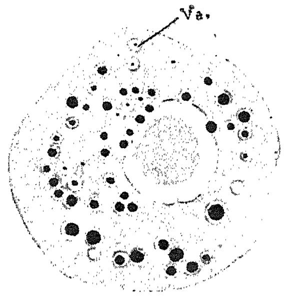

Fig. 1

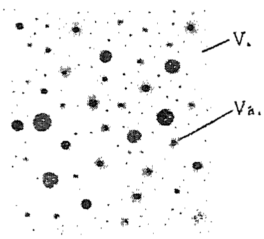

Fig. 2

as Nath and Nangia, and Hibbard and Parat claim. In addition to these vacuoles, however, there are numerous smaller granules which come to view on staining the eggs with neutral red solution from 30 to 45 minutes (Fig. 2). These two categories of vacuoles do not seem to be the different stages of the same structures but are probably fundamentally different. In the opinion of the present writer the term vacuome as originally employed by Parat is not applicable to these yolk-forming vacuoles.

The important fact which it is the aim of this paper to emphasise is, however, that the entire bulk of the proteid yolk of the egg is not formed in this way. There are unmistakable signs that a great portion of the proteid yolk arises through the direct transformation of mitochondria. Fig. 3 shows an oocyte in which the

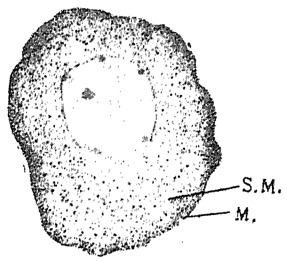

Fig. 3

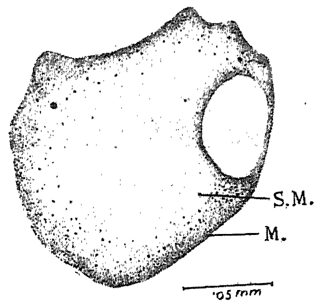

Fig. 4

mitochondria have become scattered all over. On the periphery they are more or less in a concentrated condition. Already a few mitochondrial grains are swelling. In fig. 4 a more advanced stage is represented. The mitochondria are confined almost exclusively to the periphery and the swelling of mitochondrial grains has advanced further. Ultimately these swollen grains grow into definitely recognizable

proteid yolk spherules (Fig. 5). These yolk bodies have absolutely no relationship with vacuoles of any kind. This process of mitochondrial transformation can also be observed in highly advanced eggs (Fig. 6). All intermediate stages between the

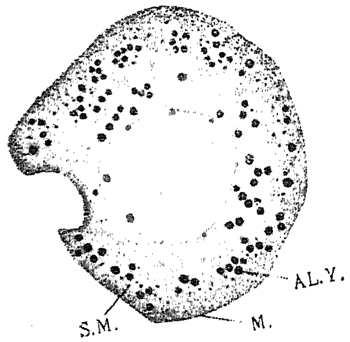

Fig. 5

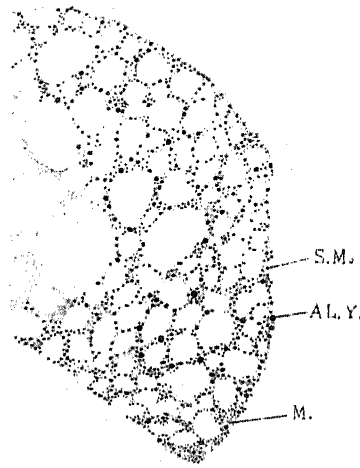

Fig. 6

slightest swollen mitochondria and the big proteid yolk spherules can be easily made out. There is no doubt that Narain's account of mitochondrial origin of yolk is quite correct. His fig. 8 bears considerable resemblance to the fig. 6 of this communication and Nath and Nangia's criticism that Narain has misinterpreted the "swollen Golgi elements as the mitochondrial yolk" is unjustified. Fig. 8 of Narain's paper (1930) has been taken from a Regaud Tupa's preparation which is not calculated to preserve the Golgi elements in an oocyte.

Both Nath and Nangia (1931) and Narain (1930 and 1937) have insisted on one particular mode of proteid yolk formation, whereas both the mitochondria and the vacuoles are directly involved in this process.

Explanation of Plates

All the diagrams have been made with camera lucida at the magnification shown by the side of fig. 4.

Fig. 1. An oocyte showing vacuoles, depositing proteid yolk inside. Bouin, methyl blue eosin.

Fig. 2. A portion of oocyte showing big and small neutral red staining bodies.

Fig. 3. A small oocyte showing granular and swollen mitochondria. F. W. A. iron alum haematoxylin.

Fig. 4. A slightly bigger oocyte showing granular and swollen mitochondria on the periphery. F. W. A. iron alum haematoxylin.

Fig. 5. An oocyte showing granular and swollen mitochondria and proteid yolk bodies. F. W. A. and iron alum haematoxylin.

Fig. 6. A part of an advanced oocyte showing fatty vacuoles, granular and swollen mitochondria and proteid yolk bodies. F. W. A. iron alum haematoxylin.

Lettering

Al. Y. = Albuminous yolk bodies.
M. = Mitochondria.
S.M. = Swollen mitochondria.
V. = Vacuome.
Va. = Vacuole.

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TABLE OF MONOMIAL SYMMETRIC FUNCTIONS OF WEIGHT 11

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Tables of monomial symmetric functions of weight up to 10 have been constructed by O'Toole,[1] Sukhatme[2] and the present authors.[3] An error in the table of 10 has since been noticed by us, and is rectified below :—

In the expansion of (1^{10}) the coefficients of $2\cdot3\cdot5$, $4\cdot6$ and $3\cdot7$ should be respectively -120960, 151200 and 172800, and not -20960, 51200 and 72800 as shown in the table.

Also in the expansion of $(2^2 1^6)$ the co-efficients of $1^2\cdot2^4$ and $1^2\cdot4^2$ should be respectively 45 and 450 instead of 135 and 360. With these corrections the table of 10 is free from errors.

The table of 11 has been constructed by us and is given below. It is believed that the table is free from all errors of computation. The method of calculation and the directions for reference have been given in our previous papers,[3] and are equally applicable here.

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1 O'Toole, Annals of Math. Statistics (U. S. A.), 2, 1931, pp. 101—149.

2 Sukhatme, P. V., Philo. Tr. R. S. (A), 760, Vol. 237, pp. 375—409.

3 Kerawala, S. M., Proc. Nat. Aca. of Sci., India, Vol. 11, 3, 1941, pp. 51—55.
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TABLE OF WEIGHT 11

Sr's→	5·6	1^4·7	1^2·27	2^2·7	1·3·7	4·7	1^3·8	1·2·8	3·8	1^2·9	2·9	1·10	11	D
(11)													1	1
(10 1)												1	—1	1
(92)											1	0	—1	1
(91^2)										1	—1	—2	2	2
(83)									1	0	0	0	—1	1
(821)								1	—1	0	—1	—1	2	1
(81^3)							1	—3	2	—3	3	6	—6	6
(74)						1	0	0	0	0	0	0	—1	1
(731)					1	—1	0	0	—1	0	0	—1	2	1
(72^2)				1	0	—1	0	0	0	0	—2	0	2	2
(721^2)			1	—1	—2	2	0	—2	2	—1	3	4	—6	2
(71^4)		1	—6	3	8	—6	—4	12	—8	12	—12	—24	24	24
(65)	1	0	0	0	0	0	0	0	0	0	0	0	—1	1

In Dictionary Order from (11) to (65). D = the Common Denominator.

TABLE OF WEIGHT 11

Sr's→	$1^3.3.5$	$1.2.3.5$	$3^2.5$	$1^2.4.5$	$2.4.5$	1.5^2	$1^5.6$	$1^3.2.6$	$1.2^2.6$	$1^2.3.6$	$2.3.6$	$1.4.6$	5.6	D
(641)												1	—1	1
(632)											1	0	—1	1
(631^2)										1	—1	—2	2	2
(62^21)									1	0	—2	—1	2	2
(621^3)								1	—3	—3	5	6	—6	6
(61^5)							1	—10	15	20	—20	—30	24	120
(5^21)						1	0	0	0	0	0	0	—2	2
(542)					1	0	0	0	0	0	0	0	—1	1
(541^2)				1	—1	—2	0	0	0	0	0	—2	4	2
(53^2)			1	0	0	0	0	0	0	0	0	0	—1	2
(5321)		1	—1	0	—1	—1	0	0	0	0	—1	0	3	1
(531^3)	1	—3	2	—3	3	6	0	0	0	—3	3	6	—12	6

In Dictionary Order from (641) to (531^3). D=the Common Denominator. (*Continued on the Next Page*).

TABLE OF WEIGHT 11

(Continued from the Preceding Page.)

Sr's→	1^4·7	1^2·2·7	2^2·7	1·3·7	4·7	1^3·8	1·2·8	3·8	1^2·9	2·9	1·10	11	D
(641)	0	0	0	0	—1	0	0	0	0	0	—1	2	1
(632)	0	0	0	0	0	0	0	—1	0	—1	0	2	1
(631^2)	0	0	0	—2	2	0	0	2	—1	1	4	—6	2
(62^21)	0	0	—1	0	1	0	—2	2	0	4	2	—6	2
(621^3)	0	—3	3	6	—6	—1	9	—8	6	—12	—18	24	6
(61^5)	—5	30	—15	—40	30	20	—60	40	—60	60	120	—120	120
(5^21)	0	0	0	0	0	0	0	0	0	0	—1	2	2
(542)	0	0	0	0	—1	0	0	0	0	—1	0	2	1
(541^2)	0	0	0	0	2	0	0	0	—1	1	4	—6	2
(53^2)	0	0	0	0	0	0	0	—2	0	0	0	2	2
(5321)	0	0	0	—1	1	0	—1	3	0	2	2	—6	1
(531^3)	0	0	0	6	—6	—1	3	—8	6	—6	—18	24	6

In Dictionary Order from (641) to (531^3). D=the Common Denominator.

TABLE OF WEIGHT 11

S_r's→	$1^4{\cdot}3{\cdot}4$	$1^2{\cdot}2{\cdot}3{\cdot}4$	$2^2{\cdot}3{\cdot}4$	$1{\cdot}3^2{\cdot}4$	$1^3{\cdot}4^2$	$1{\cdot}2{\cdot}4^2$	$3{\cdot}4^2$	$1^6{\cdot}5$	$1^4{\cdot}2{\cdot}5$	$1^2{\cdot}2^2{\cdot}5$	$2^3{\cdot}5$	$1^3{\cdot}3{\cdot}5$	$1{\cdot}2{\cdot}3{\cdot}5$	D
(52^3)											1	0	0	6
(52^21^2)										1	—1	0	—4	4
(521^4)									1	—6	3	—4	20	24
(51^6)								1	—15	45	—15	40	—120	720
(4^23)							1	0	0	0	0	0	0	2
(4^221)						1	—1	0	0	0	0	0	0	2
(4^21^3)					1	—3	2	0	0	0	0	0	0	12
(43^21)				1	0	0	—2	0	0	0	0	0	0	2
(432^2)			1	0	0	0	—1	0	0	0	0	0	0	2
(4321^2)		1	—1	—2	0	—2	4	0	0	0	0	0	—2	2
(431^4)	1	—6	3	8	—4	12	—14	0	0	0	0	—4	12	24

In Dictionary Order from (52^3) to (431^4). D=the Common Denominator. (*Continued on the Next Page.*)

TABLE OF WEIGHT 11

(Continued from the Preceding Page.)

Sr's→	$3^2.5$	$1^2.4.5$	$2.4.5$	1.5^2	$1^5.6$	$1^3.2.6$	$1.2^2.6$	$1^2.3.6$	$2.3.6$	$1.4.6$	5.6	$1^4.7$	D
(52^3)	0	0	—3	0	0	0	0	0	0	0	2	0	6
(52^21^2)	2	—1	5	4	0	0	—2	0	4	2	—10	0	4
(521^4)	—8	12	—18	—24	0	—4	12	12	—20	—24	48	—1	24
(51^6)	40	—90	90	144	—6	60	—90	—120	120	180	—264	30	720
(4^23)	0	0	0	0	0	0	0	0	0	0	0	0	2
(4^221)	0	0	—2	0	0	0	0	0	0	—2	2	0	2
(4^21^3)	0	—6	6	6	0	0	0	0	0	12	—12	0	12
(43^21)	—1	0	0	0	0	0	0	0	0	—1	1	0	2
(432^2)	0	0	—2	0	0	0	0	0	—2	0	2	0	2
(4321^2)	2	—1	5	2	0	0	0	—1	3	6	—8	0	2
(431^4)	—8	24	—24	—24	0	0	0	12	—12	—48	48	—1	24

In Dictionary Order from (52^3) to (431^4). D=the Common Denominator.

Table of Weight 11

Sr's→	$1^2.2.7$	$2^2.7$	$1.3.7$	4.7	$1^3.8$	$1.2.8$	3.8	$1^2.9$	2.9	1.10	11	D
(52^3)	0	—3	0	3	0	0	0	0	6	0	—6	6
(52^21^2)	—2	4	4	—6	0	8	—8	2	—14	—12	24	4
(521^4)	18	—15	—32	30	8	—48	40	—36	60	96	—120	24
(51^6)	—180	90	240	—180	—120	360	—240	360	—360	—720	720	720
(4^23)	0	0	0	—2	0	0	—1	0	0	0	2	2
(4^221)	0	0	0	4	0	—1	1	0	2	2	—6	2
(4^21^3)	0	0	0	—12	—1	3	—2	6	—6	—18	24	12
(43^21)	0	0	—2	4	0	0	4	0	0	2	—6	2
(432^2)	0	—1	0	3	0	0	2	0	4	0	—6	2
(4321^2)	—1	1	6	—12	0	4	—10	2	—8	—12	24	2
(431^4)	6	—3	—32	54	8	—24	40	—36	36	96	—120	24

In Dictionary Order from (52^3) to (431^4). D = the Common Denominator. *(Continued on the Next Page)*

TABLE OF WEIGHT 11

(*Continued from the Preceding Page.*)

Sr's→	$2^4.3$	$1^5.3^2$	$1^3.2.3^2$	$1.2^2.3^2$	$1^2.3^3$	2.3^3	$1^7.4$	$1^5.2.4$	$1^3.2^2.4$	$1.2^3.4$	$1^4.3.4$	$1^2.2.3.4$	$2^2.3.4$	D
(42^31)										1	0	0	—3	6
(42^21^3)									1	—3	0	—6	8	12
(421^5)								1	—10	15	—5	50	—35	120
(41^7)							1	—21	105	—105	70	—420	210	5040
(3^32)						1	0	0	0	0	0	0	0	6
(3^31^2)					1	—1	0	0	0	0	0	0	0	12
(3^22^21)				1	0	—2	0	0	0	0	0	0	—2	4
(3^221^3)			1	—3	—3	5	0	0	0	0	0	—6	6	12
(3^21^5)		1	—10	15	20	—20	0	0	0	0	—10	60	—30	240
(32^4)	1	0	0	0	0	0	0	0	0	0	0	0	—6	24

In Dictionary Order from (42^31) to (32^4). D = the Common Denominator.

TABLE OF WEIGHT 11

$Sr's \rightarrow$	$1\cdot3^2\cdot4$	$1^3\cdot4^2$	$1\cdot2\cdot4^2$	$3\cdot4^2$	$1^6\cdot5$	$1^4\cdot2\cdot5$	$1^2\cdot2^2\cdot5$	$2^3\cdot5$	$1^3\cdot3\cdot5$	$1\cdot2\cdot3\cdot5$	$3^2\cdot5$	$1^2\cdot4\cdot5$	D
(42^31)	0	0	—3	3	0	0	0	—1	0	0	0	0	6
(42^21^3)	6	—1	15	—14	0	0	—3	3	0	12	—6	9	12
(421^5)	—40	20	—90	70	0	—5	30	—15	20	—100	40	—120	120
(41^7)	280	—210	630	—420	—7	105	—315	105	—280	840	—280	1134	5040
(3^32)	0	0	0	0	0	0	0	0	0	0	—3	0	6
(3^31^2)	—6	0	0	6	0	0	0	0	0	0	6	0	12
(3^22^21)	—1	0	0	2	0	0	0	0	0	—4	6	0	4
(3^221^3)	18	0	6	—18	0	0	0	0	—2	18	—22	6	12
(3^21^5)	—110	20	—60	100	0	0	0	0	40	—120	104	—120	240
(32^4)	0	0	0	3	0	0	0	—4	0	0	0	0	24

In Dictionary Order from (42^31) to (32^4). D=the Common Denominator. (*Continued on the Next Page*)

TABLE OF WEIGHT 11

(Continued from the Preceding Page.)

Sr's→	2·4·5	1·5²	1⁵·6	1³·2·6	1·2²·6	1²·3·6	2·3·6	1·4·6	5·6	1⁴·7	1²·2·7	D
(42^31)	9	0	0	0	—3	0	6	5	—8	0	0	6
(42^21^3)	—33	—12	0	—2	12	6	—22	—36	42	0	12	12
(421^5)	174	120	—1	30	—75	—80	120	270	—264	10	—120	120
(41^7)	—1134	—1008	42	—420	630	840	—840	—2100	1848	—210	1260	5040
(3^32)	0	0	0	0	0	0	—3	0	3	0	0	6
(3^31^2)	0	0	0	0	0	—3	3	6	—6	0	0	12
(3^22^21)	4	2	0	0	—1	0	10	1	—10	0	0	4
(3^221^3)	—18	—12	0	—1	3	15	—29	—30	42	0	6	12
(3^21^5)	120	120	—1	10	—15	—140	140	270	—264	10	—60	240
(32^4)	12	0	0	0	0	0	8	0	—8	0	0	24

In Dictionary Order from (42^31) to (32^4). D = the Common Denominator.

TABLE OF WEIGHT 11

Sr's→	$2^2.7$	$1.3.7$	4.7	$1^3.8$	$1.2.8$	3.8	$1^2.9$	2.9	1.10	11	D
(42^31)	6	0	—12	0	6	—6	0	—18	—6	24	6
(42^21^3)	—18	—24	54	2	—42	40	—18	66	72	—120	12
(421^5)	90	200	—300	—60	300	—240	240	—360	—600	720	120
(41^7)	—630	—1680	1980	840	—2520	1680	—2520	2520	5040	—5040	5040
(3^32)	0	0	0	0	0	6	0	2	0	—6	6
(3^31^2)	0	12	—12	0	0	—18	2	—2	—12	24	12
(3^22^21)	2	4	—6	0	4	—16	0	—12	—6	24	4
(3^221^3)	—6	—48	48	2	—24	70	—18	42	72	—120	12
(3^21^5)	30	320	—300	—60	180	—360	240	—240	—600	720	240
(32^4)	12	0	—12	0	0	—6	0	—24	0	24	24

In Dictionary Order from (42^31) to (32^4). D=the Common Denominator. (*Continued on the Next Page.*)

TABLE OF WEIGHT 11

(*Continued from the Preceding Page.*)

S_r's→	1^{11}	$1^9.2$	$1^7.2^2$	$1^5.2^3$	$1^3.2^4$	1.2^5	$1^8.3$	$1^6.2.3$	$1^4.2^2.3$	$1^2.2^3.3$	$2^4.3$	$1^5.3^2$	$1^3.2.3^2$	D
(32^31^2)										1	—1	0	0	12
(32^21^4)									1	—6	3	0	—8	48
(321^6)								1	—15	45	—15	—6	100	720
(31^8)							1	—28	210	—420	105	112	—1120	40320
(2^51)						1	0	0	0	0	—5	0	0	120
(2^41^3)					1	—3	0	0	0	—12	14	0	0	144
(2^31^5)				1	—10	15	0	0	—15	110	—65	0	60	720
(2^21^7)			1	—21	105	—105	0	—14	280	—1050	420	42	—980	10080
(21^9)		1	—36	378	—1260	945	—9	420	—4410	11340	—3465	—1008	13440	362880
(1^{11})	1	—55	990	—6930	17325	—10395	330	—9240	69300	—138600	34650	18480	—184800	39916800

In Dictionary Order from (32^31^2) to (1^{11}). D = the Common Denominator.

TABLE OF WEIGHT 11

Sr's→	$1\cdot2^2\cdot3^2$	$1^2\cdot3^3$	$2\cdot3^3$	$1^7\cdot4$	$1^5\cdot2\cdot4$	$1^3\cdot2^2\cdot4$	$1\cdot2^3\cdot4$	$1^4\cdot3\cdot4$	$1^2\cdot2\cdot3\cdot4$	$2^2\cdot3\cdot4$	$1\cdot3^2\cdot4$	$1^3\cdot4^2$	D
(32^31^2)	—6	0	6	0	0	0	—2	0	—3	15	6	0	12
(32^21^4)	32	12	—28	0	0	—4	12	—1	54	—65	—80	4	48
(321^6)	—210	—120	160	0	—6	60	—90	60	—570	390	660	—120	720
(31^8)	1680	1120	—1120	—8	168	—840	840	—980	5880	—2940	—5600	1680	40320
(2^51)	0	0	0	0	0	0	—10	0	0	30	0	0	120
(2^41^3)	36	0	—24	0	0	—6	42	0	36	—120	—36	3	144
(2^31^5)	—300	—60	180	0	—3	90	—255	15	—510	675	480	—60	720
(2^21^7)	2590	840	—1400	—1	105	—1155	1995	—490	5880	—4830	—4900	1050	10080
$(2\ 1^9)$	—25200	—10080	12320	72	—2268	15120	—18900	8820	—68040	41580	50400	—15120	362880
(1^{11})	277200	123200	—123200	—1980	41580	—207900	207900	—138600	831600	—415800	—554400	207900	39916800

In Dictionary Order from (32^31^2) to (1^{11}). D=the Common Denominator. (*Continued on the Next Page.*)

TABLE OF WEIGHT 11

(Continued from the Preceding Page.)

Sr's→	$1{\cdot}2{\cdot}4^2$	$3{\cdot}4^2$	$1^6{\cdot}5$	$1^4{\cdot}2{\cdot}5$	$1^2{\cdot}2^2{\cdot}5$	$2^3{\cdot}5$	$1^3{\cdot}3{\cdot}5$	$1{\cdot}2{\cdot}3{\cdot}5$	$3^2{\cdot}5$	$1^2{\cdot}4{\cdot}5$	$2{\cdot}4{\cdot}5$	D
(32^31^2)	6	—12	0	0	—3	5	0	24	—18	3	—33	12
(32^21^4)	—60	86	0	—2	24	—18	16	—160	104	—60	168	48
$(32\,1^6)$	540	—600	—1	45	—225	105	—280	1224	—664	810	—1134	720
(31^8)	—5040	4620	56	—840	2520	—840	3584	—10752	4928	—9072	9072	40320
(2^51)	15	—15	0	0	0	20	0	0	0	0	—60	120
(2^41^3)	—81	78	0	0	36	—60	0	—144	72	—36	252	144
(2^31^5)	630	—570	0	30	—360	294	—120	1320	—600	540	—1512	720
(2^21^7)	—5670	4620	14	—630	3654	—1974	2240	—12096	4928	—6804	11340	10080
(21^9)	56700	—41580	—504	10584	—40824	16632	—32256	120960	—44352	81648	—99792	362880
(1^{11})	—623700	415800	11088	—166320	498960	—166320	443520	—1330560	443520	—997920	997920	39916800

In Dictionary Order from (32^31^2) to (1^{11}). D=the Common Denominator.

TABLE OF WEIGHT 11

Sr's→	1.5^2	$1^5.6$	$1^3.2.6$	$1.2^2.6$	$1^2.3.6$	$2.3.6$	$1.4.6$	5.6	$1^4.7$	$1^2.2.7$	D
(32^31^2)	−12	0	0	12	2	−44	−16	46	0	6	12
(32^21^4)	96	0	16	−72	−84	212	192	−264	2	−84	48
(321^6)	−864	12	−24	540	960	−1320	−1800	1848	−90	900	720
(31^8)	8064	−336	3360	−5040	−10080	10080	16800	−14784	1680	−10080	40320
(2^51)	0	0	0	20	0	−40	−20	40	0	0	120
(2^41^3)	72	0	8	−132	−24	256	156	−264	0	−72	144
(2^31^5)	−720	2	−200	930	580	−1660	−1500	1848	−30	900	720
(2^21^7)	7056	−126	2940	−7770	−7560	12600	14700	−14784	840	−10080	10080
(21^9)	−72576	3024	−40320	75600	90720	−110880	−151200	133056	−15120	116640	362880
(1^{11})	798336	−55440	554400	−831600	−1108800	1108800	1663200	−1330560	237600	−1425600	39916800

In Dictionary Order from (32^31^2) to (1^{11}) D=the Common Denominator. (*Continued on the Next Page.*)

TABLE OF WEIGHT 11

(Continued from the Preceding Page.)

Sr's→	$2^2.7$	$1.3.7$	4.7	$1^3.8$	$1.2.8$	3.8	$1^2.9$	2.9	1.10	11	D
(3^231^2)	—24	—24	42	0	—36	60	—6	78	48	—120	12
(32^21^4)	102	256	—276	—24	264	—360	144	—384	—480	720	48
(321^6)	—630	—2160	1980	480	—2160	2400	—1800	2520	4320	—5040	720
(31^8)	5040	19200	—15840	—6720	20160	—18480	20160	—20160	—40320	40320	40320
(2^51)	—60	0	60	0	—30	30	0	120	24	—120	120
(2^41^3)	216	144	—288	—6	306	—300	72	—552	—360	720	144
(2^31^5)	—1170	—1680	1980	240	—2520	2280	—1200	3360	3600	—5040	720
(2^21^7)	8280	16800	—15840	—4200	22680	—18480	15120	—25200	—35280	40320	10080
(21^9)	—71280	—172800	142560	60480	—226800	166320	—181440	221760	362880	—362880	362880
(1^{11})	712800	1900800	—1425600	—831600	2494800	—1663200	2217600	—2217600	—3991680	3628800	39916800

In Dictionary Order from (3^231^2) to (1^{11}) D=the Common Denominator.

A NOTE ON THE PSEUDO-RECIPROCAL IN THE WAVE-TENSOR CALCULUS

By Brij Nath

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In this note we discuss a few properties of the *pseudo-reciprocal* which is defined in Eddington's wave-tensor calculus* as follows :—

If S is a non-singular E-number, then it is possible to determine a unique E-number R such that $RS=1$. R is said to be a reciprocal of S. If S is singular, it is possible to obtain an infinitude of solutions of $RS=0$. Any such solution R is called a pseudo-reciprocal of S.

The well-known properties of the reciprocal and the pseudo-reciprocal are :—

(1) Any E-number S commutes with its reciprocal.

(2) Any E-number which commutes with an E-number S commutes also with its reciprocal.

(3) A pseudo-reciprocal is necessarily singular.

(4) If R is a pseudo-reciprocal of S, XR is also a pseudo-reciprocal of S, X being any E-number.

In this paper we will be concerned with the following results :—

(*a*) A pseudo-reciprocal of a singular E-number S does not necessarily commute with S. Or if R is a pseudo-reciprocal of S, S is not necessarily a pseudo-reciprocal of R. This should be compared with (1) above for a reciprocal. We will determine the conditions under which a singular E-number commutes with its pseudo-reciprocal.

(*b*) If R is a pseudo-reciprocal of S, XR is also a pseudo-reciprocal of S, X being any E-number. But RX is not necessarily a pseudo-reciprocal. We will determine the conditions such that RX is also a pseudo-reciprocal of S.

For reasons of mathematical simplicity we shall work with a two dimensional case of the wave-tensor calculus. The complete orthogonal set consists of 4 E-symbols E_1, E_2, E_3 and E_4 satisfying the relations :—

$$E_\mu^2 = -1, \qquad (\mu=1, 2, 3, 4)$$

$$E_\mu E_\nu = -E_\nu E_\mu, \qquad (\mu\neq\nu\neq4)$$

$$E_i E_j = E_k. \qquad (i\neq j\neq k\neq4 \text{ and } i, j, k \text{ are in cyclic order})$$

$E_4 \equiv i$ (not to be confused with the suffix).

* See Eddington's "The relativity theory of electron and proton" 1 (1936), Chap. II, for explanation of notation.

Let $R = r_1E_1 + r_2E_2 + r_3E_3 + r_4E_4$.

$S = s_1E_1 + s_2E_2 + s_3E_3 + s_4E_4$.

As $RS = 0$, we have

$$r_2s_3 - r_3s_2 + ir_1s_4 + ir_4s_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$r_3s_1 - r_1s_3 + ir_2s_4 + ir_4s_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$r_1s_2 - r_2s_1 + ir_3s_4 + ir_4s_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$r_1s_1 + r_2s_2 + r_3s_3 + r_4s_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating r_1, r_2, r_3 and r_4, we have

$$\begin{vmatrix} is_4 & s_3 & -s_2 & is_1 \\ -s_3 & is_4 & s_1 & is_2 \\ s_2 & -s_1 & is_4 & is_3 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix} = 0$$

or $\quad (s_1^2 + s_2^2 + s_3^2 - s_4^2)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

It can be easily verified that on account of (5) every minor of the 3rd order in the above determinant is zero. The matrix of the determinant is, therefore, of rank 2. This means that out of the four co-efficients r_1, r_2, r_3 and r_4 of the E-number R two can be chosen arbitrarily.

Solving (1) and (2), we have

$$r_1 = \frac{r_3(s_1s_3 + is_2s_4) + r_4(s_1s_4 + is_2s_3)}{s_3^2 - s_3^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$r_2 = \frac{r_3(s_2s_3 - is_1s_4) + r_4(s_2s_4 - is_1s_3)}{s_3^2 - s_4^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If $SR = 0$, we should have the four equations :—

$$s_2r_3 - s_3r_2 + is_1r_4 + is_4r_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$s_3r_1 - s_1r_3 + is_2r_4 + is_4r_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$s_1r_2 - s_2r_1 + is_3r_4 + is_4r_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$s_1r_1 + s_2r_2 + s_3r_3 + s_4r_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Substituting the values of r_1 and r_2 in (8), (9) and (10) the equations are not satisfied. Hence, $SR \neq 0$.

If SR is equal to zero, equations (8), (9) and (10) are also true. Subtracting (8), (9) and (10) from (1), (2) and (3) respectively, we get $\frac{r_1}{s_1} = \frac{r_2}{s_2} = \frac{r_3}{s_3} = \frac{-r_4}{s_4}$.

This is the condition which must be satisfied in order that S and its pseudo-reciprocal R may commute.

By a proof similar to the above it can be shown that if R is a pseudo-reciprocal of S, RX will also be a pseudo-reciprocal of S if X is a pseudo-reciprocal of S.

RADIAL OSCILLATIONS OF A VARIABLE STAR

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SUMMARY

In a recent and important paper Prof. A. C. Banerji has shown that no radial mode of oscillation of large amplitude is possible for a variable star. Adopting Prof. Banerji's method it has been shown in this paper that, if the density vary inversely as an integral power of the distance from the centre, no radial mode of oscillation is possible except for Dr. Sterne's model of the sphere with a vanishing core in which the density varies inversely as the square of the distance from the centre.

The pulsation theory of Cepheid Variables, advanced by Shapley[17] and Plummer,[14] has been mathematically developed by Eddington.[4] Eddington has obtained the period of the zero'th mode and Edgar[11] the period of the first mode corresponding to the density distribution of the Emden polytrope of index 3. Sterne[18] has considered the modes of radial oscillation of the following three stellar models :

1. The sphere of uniform density.
2. The sphere in which the density varies inversely as the square of the distance to the centre.
3. The sphere in which nearly all the mass is contained in a particle at the centre.

Shinjo and Jeans[13] have objected to the pulsation theory of Cepheids on the ground that the velocity-curves would in that theory contain higher modes of incommensurable period which have not been observed. Eddington[5] on the basis of Woltje's calculations concludes that the dissipation is relatively greater for the overtones, so that their absence, in his opinion, is accounted for. Sterne,[18] however, is of the opinion that a few pulsating stars might be expected to show evidence of higher modes superimposed on the zero'th. He[19] and Schwartzschild[15] have recently observed overtone pulsations for the short-period variables, and Schwartzschild,[16] in a very recent paper, has calculated the period of the fundamental mode and the first four spherically symmetrical overtone pulsations for the standard model for four different values of the ratio of specific heats.

All these investigations, however, do not take account of the square of the amplitude. In his investigations, Eddington[4] has introduced the following simplifying assumptions :

(a) The oscillations are adiabatic ;

(b) the oscillations are radial, that is, the displacement at any point is a function only of the distance of the point from the centre and of the time; and

(c) the amplitude of the oscillations is small, so that its square may be neglected.

The assumption (a) evidently fails near the surface of the star. Eddington has given justification that this does not affect the orders of the magnitudes concerned. In fact he remarks[6] that the adiabatic approximation is much more accurate for a Cepheid than for ordinary sound waves. The assumption (b) may be justified on the ground that a radial oscillation, as it involves only radiative viscosity, is easier to maintain than an oscillation of the most general type which involves material viscosity as well.[15] The assumption (c), however, cannot be justified, as in typical Cepheids the surface amplitude may amount to one-twelfths of the radius, as Eddington[7] himself admits, and the square of the variation in pressure to five times this amount. Eddington has shown[10] that the retention of the second order terms will give "a velocity-curve having the general characteristics of the observed velocity-curves of Cepheids, *viz.*, a sharp decrease from maximum to minimum receding velocity and a slower return to maximum with indications of a hump in the curve."[7]

In a recent and very interesting paper, A. C. Banerji[1] has investigated the modes of radial oscillation of the following stellar models :

1. The sphere of uniform density.
2. The sphere with a small, homogeneous central core and an annulus where the density at any point varies inversely as the p'th power of the distance of the point from the centre, where p is any positive integer (excluding 1 and 3).

Banerji has shown that the oscillations become unstable for large amplitude, and, on this basis, has given an entirely new theory of the origin of the solar system.

In what follows, we will show that the radial oscillations of Banerji's model 2 are unstable also for small amplitude. We will further show that for a sphere with a vanishingly small core and density varying inversely as an integral power of the distance from the centre, radial oscillations are only possible for the inverse square law of density, that is, Sterne's model 2.

Eddington[4] has derived the fundamental equation of adiabatic pulsation, retaining only the first power of the amplitude.

The equation of oscillatory motion is

$$\ddot{\xi} = -\frac{g_0\xi_0^2}{\xi^2} - \frac{1}{\rho}\frac{dP}{d\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ξ, g, ρ and P denote distance from the centre, gravity, density and pressure respectively. We shall use the suffix zero to indicate the undisturbed values of these quantities. Let the displacements ξ_1, ρ_1 and P_1 be given by

$$\xi=\xi_0(1+\xi_1)\,;\ \rho=\rho_0(1+\rho_1)\,;\ P=P_0(1+P_1) \quad . \quad . \quad . \quad . \quad (2)$$

For small displacements, the motion is simple harmonic and we have

$$\xi_1=a_1\cos nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where a_1 is the amplitude at ξ_0.

The equation of conservation of mass is

$$\rho\xi^2 d\xi=\rho_0\xi_0{}^2 d\xi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For adiabatic oscillations, we have

$$dP/P=\gamma\alpha\rho/\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where γ is the effective ratio of specific heats (regarding the matter and enclosed radiation as one system).[8]

Neglecting higher powers of ξ_1 than the first, Eddington[9] has shown that the differential equation (1) reduces to the equation :

$$\frac{d^2a_1}{d\xi_0{}^2}+\frac{4-\nu}{\xi_0}\frac{da_1}{d\xi_0}+\left[\frac{n^2\rho_0}{P_0\gamma}-\frac{\alpha\nu}{\xi_0{}^2}\right]a_1=0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$\nu=\frac{g_0\rho_0\xi_0}{P_0} \quad \text{and} \quad \alpha=3-\frac{4}{\gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Changing the independent variable in (6) from ξ_0 to x by the substitution $\xi_0=Rx$, the equation (6) becomes

$$\frac{d^2a_1}{dx^2}+\frac{4-\nu}{x}\frac{da_1}{dx}+\frac{R\rho_0}{P_0}\left(\frac{n^2R}{\gamma}-\frac{\alpha g_0}{x}\right)a_1=0 \quad , \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where R is the radius of the star.

We will first consider the small oscillations of Banerji's model of a sphere with a small, homogeneous central core and density in the annulus varying inversely as an integral power of the distance from the centre.

Let ρ_a be the undisturbed uniform density of a small core of radius

$$a=\mu R\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where R is the radius of the star. Also let

$$\rho_0=K/\xi_0^p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where ρ_0 is the undisturbed density at a point in the annulus distant ξ_0 from the centre ($a\leq\xi_0\leq R$), K is a constant, and let $\bar{\rho}$ be the mean density in the annulus. We will also suppose that p in (10) is any positive integer with the exception of

1 and 3, as these latter values make g_0 and P_0 indeterminate as we shall see later on in equations (16) and (18).

We have

$$\rho_a = K/a^p = K/R^p \mu^p \quad . \quad . \quad . \quad . \quad (11)$$

and

$$\frac{4\pi}{3}\bar{\rho}\,(R^3 - a^3) = \frac{4\pi K}{p-3}\left[\frac{1}{a^{p-3}} - \frac{1}{R^{p-3}}\right] \quad . \quad . \quad . \quad . \quad (12)$$

From (9), (11) and (12), we have

$$K = \frac{\bar{\rho}\,(p-3)\mu^{p-3}(1-\mu^3)R^p}{3(1-\mu^{p-3})} \quad . \quad . \quad . \quad . \quad (13)$$

and

$$\bar{\rho} = \frac{3\rho_a(1-\mu^{p-3})\mu^3}{(p-3)(1-\mu^3)} \quad . \quad . \quad . \quad . \quad (14)$$

The undisturbed value of gravity, g_0, at a point ξ_0 is given by

$$g_0 = \frac{\frac{4}{3}\pi\rho_a a^3 G + \frac{4\pi K}{p-3}\left[\frac{1}{a^{p-3}} - \frac{1}{\xi_0{}^{p-3}}\right]G}{\xi_0{}^2}, \quad . \quad . \quad . \quad . \quad (15)$$

where G is the gravitational constant.

From (11) and (15), we have

$$g_0 = \frac{4\pi KG}{3(p-3)}\frac{1}{\xi_0{}^2}\left(\frac{p}{a^{p-3}} - \frac{3}{\xi_0{}^{p-3}}\right)$$

$$= \frac{4\pi KG}{3(p-3)R^{p-1}x^2}\left(\frac{p}{\mu^{p-3}} - \frac{3}{x^{p-3}}\right), \quad . \quad . \quad . \quad (16)$$

where

$$\xi_0 = Rx(\mu \leq x \leq 1) \quad . \quad . \quad . \quad . \quad (17)$$

Integrating the hydrostatic equation of equilibrium in the undisturbed state, *viz.*,

$$\frac{dP_0}{d\xi_0} = -g_0\rho_0,$$

we have, for the pressure P_0 at ξ_0,

$$P_0 = \frac{4\pi K^2 G}{3(p-3)R^{2p-2}}\left\{\frac{p}{(p+1)\mu^{p-3}}\left(\frac{1}{x^{p+1}} - 1\right) - \frac{3}{2p-2}\left(\frac{1}{x^{2p-2}} - 1\right)\right\}. \quad (18)$$

assuming that the pressure vanishes on the boundary.

Hence, we have

$$\frac{g_0\rho_0\xi_0}{P_0} = \frac{(p+1)(2p-2)(px^{p-3} - 3\mu^{p-3})}{p(2p-2)x^{p-3}(1-x^{p+1}) - 3(p+1)\mu^{p-3}(1-x^{2p-2})} \quad . \quad . \quad (19)$$

With the substitutions (16), (18) and (19), and, after simplification, we can write (8) in the form

$$\Big[-3(p+1)\mu^{p-3}x^2+p(2p-2)x^{p-1}-\Big\{p(2p-2)-3(p+1)\mu^{p-3}\Big\}x^{2p}\Big]\frac{d^2a_1}{dx^2}$$
$$+\Big[6(p+1)(p-3)\mu^{p-3}x+(3-p)p(2p-2)x^{p-2}$$
$$-4\Big\{p(2p-2)-3(p+1)\mu^{p-3}\Big\}x^{2p-4}\Big]\frac{da_1}{dx}$$
$$+\Big[3\alpha(p+1)(2p-2)\mu^{p-3}-\alpha p(p+1)(2p-2)x^{p-3}+fx^p\Big]a_1$$
$$=0, \qquad (20)$$

where
$$f=\frac{9n^2}{2\pi G\overline{\rho\gamma}}\frac{(p^2-1)(1-\mu^{p-3})}{1-\mu^3}. \qquad (21)$$

(20) is a differential equation of the second order, having regular singularities[22] at $x=0$ and $x=1$.

We assume for the solution of (20) the following series

$$a_1=x^q\sum_0^\infty b_\lambda x^\lambda. \qquad (22)$$

The indicial equation[22] gives two real roots

$$q=\frac{(2p-5)\pm\sqrt{(2p-5)^2+8\alpha(p-1)}}{2} \qquad (23)$$

For either root we have the following recurrence formula:

$$fb_{\lambda-p}-\Big\{p(2p-2)-3(p+1)\mu^{p-3}\Big\}(\lambda+q-2p+2)(\lambda+q-2p+5)b_{\lambda-2p+2}$$
$$+p(2p-2)\Big\{(\lambda+q-p+3)(\lambda+q-2p+5)-\alpha(p+1)\Big\}b_{\lambda-p+3}$$
$$+3(p+1)\mu^{p-3}\Big\{\alpha(2p-2)-(\lambda+q)(\lambda+q-2p+5)\Big\}b_\lambda=0 \qquad (24)$$

The differential equation (20) has been obtained by Banerji[1] in his investigation and the series solution (22) proved to be divergent on the surface. We will give the following alternative investigation of the convergence of (22).

Dividing (24) by λ^2 and proceeding to the limit as $\lambda\to\infty$, we have

$$p(2p-2)\Big\{1-\mathrm{Lt}\frac{b_{\lambda-p+3}}{b_{\lambda-2p+2}}\Big\}=3(p+1)\mu^{p-3}\Big\{1-\mathrm{Lt}\frac{b_\lambda}{b_{\lambda-2p+2}}\Big\} \qquad (25)$$

Let
$$\mathrm{Lt}\frac{b_\lambda}{b_{\lambda-1}}=l \qquad (26)$$

Then we have

$$\mathrm{Lt}\frac{b_{\lambda-p+3}}{b_{\lambda-2p+2}}=l^{p+1}\text{, and }\mathrm{Lt}\frac{b_{\lambda}}{b_{\lambda-2p+2}}=l^{2p-2} \quad \ldots \quad (27)$$

From (25) and (27), we have

$$p(2p-2)(1-l^{p+1})-3(p+1)\mu^{p-3}(1-l^{2p-2})=0. \quad \ldots \quad (28)$$

From (26) and (28), we have

$$\mathrm{Lt}\frac{b_\lambda}{b_{\lambda-1}}=l=1 \quad \ldots \quad (29)$$

Hence, the series solution for (20) has unit radius of convergence. This also follows from the general theory of linear differential equations having regular singularities.[1,2] The equation (20) has regular singularities at $x=0$ and $x=1$. The series solution (22) is convergent in the neighbourhood of the origin right up to the next singularity ($x=1$).

We proceed to test the convergence of (22) for $x=1$.

From (29) we have all the terms ultimately to be of the same sign, and

$$\frac{b_\lambda}{b_{\lambda-1}}=1-\varepsilon\ , \quad \ldots \quad (30)$$

where ε is a function of λ such that

$$\varepsilon=0\left(\frac{1}{\lambda^p}\right) \quad \ldots \quad (31)$$

p being positive.[2,3]

Then we have

$$\frac{b_{\lambda-p+3}}{b_{\lambda-2p+2}}=(1-\varepsilon)^{p+1}=1-(p+1)\varepsilon\ , \quad \ldots \quad (32)$$

and

$$\frac{b_\lambda}{b_{\lambda-2p+2}}=(1-\varepsilon)^{2p-2}=1-(2p-2)\varepsilon \quad \ldots \quad (33)$$

to the first power of ε.

We have, from the recurrence formula (24):

$$\begin{aligned}f\frac{b_{\lambda-p}}{b_{\lambda-2p+2}} &= \lambda^2\Big[\Big\{p(2p+2)-3(p+1)\mu^{p-3}\Big\}-p(2p-2)\frac{b_{\lambda-p+3}}{b_{\lambda-2p+2}} \\ &\qquad +3(p+1)\mu^{p-3}\frac{b_\lambda}{b_{\lambda-2p+2}}\Big] \\ &+\lambda\Big[\Big\{p(2p-2)-3(p+1)\mu^{p-3}\Big\}(2q-4p+7)-p(2p-2)(2q-3p+8) \\ &\qquad \frac{b_{\lambda-p+3}}{b_{\lambda-2p+2}}+3(p+1)\mu^{p-3}(2q+5-2p)\frac{b_\lambda}{b_{\lambda-2p+2}}\Big]\end{aligned}$$

$$+\Bigg[\Big\{p(2p-2)-3(p+1)\mu^{p-3}\Big\}(q-2p+2)(q-2p+5)$$
$$-p(2p-2)\Big\{(q-p+3)(q-2p+5)-\alpha(p+1)\Big\}\frac{b_{\lambda-p+3}}{b_{\lambda-2p+2}}$$
$$+3(p+1)\mu^{p-3}\Big\{q(q-2p+5)-\alpha(2p-2)\Big\}\frac{b_\lambda}{b_{\lambda-2p+2}}\Bigg] \quad . \quad . \quad (34)$$

From (32), (33) and (34), we have, keeping terms of the highest order,

$$(p-3\mu^{p-3})\,\varepsilon\lambda^2=(p-3\mu^{p-3})\lambda,$$

whence we have

$$\varepsilon=\frac{1}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Hence, from (30), we have

$$\frac{b_\lambda}{b_{\lambda-1}}=1-\frac{1}{\lambda}+0\left(\frac{1}{\lambda^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

if we do not neglect higher powers of ε than the first.

(36) shows that the series soloution (22) is divergent for $x=1$. (Gauss's rule[2].)

Hence, we have, by an extension of Abel's theorem to series divergent on the circle of convergence[3]

$$\mathrm{Lt}_{x\to1}\sum_0^\infty b_\lambda x^\lambda=\sum_0^\infty b_\lambda=\infty \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Thus the amplitude of the oscillations increases without limit as we approach the surface of the star, and the model in question is therefore unstable for radial oscillations.

It is interesting to note that Sterne's models 1 and 2, *viz*., the sphere of uniform density and the sphere in which the density varies inversely as the square of the distance from the centre, are the limiting cases of the model just considered when the core is made evanescent, that is, as $\mu\to0$ and $p=0$ or 2. It can be easily verified that the differential equation (20) reduces to the corresponding ones of Sterne, leading to his results.[18]

The differential equation (20) enables us to investigate a generalisation of Sterne's model 2, *viz*., the sphere with an evanescent core in which the density varies inversely as the pth power of the distance from the centre ($p>3$).

Making $\mu\to0$ in (20), we have, for $p>3$,

$$\left(1-x^{p+1}\right)\frac{d^2a_1}{dx^2}-\frac{1}{x}\Big\{(p-3)+4x^{p+1}\Big\}\frac{da_1}{dx}+\Big\{fx-\frac{\alpha(p+1)}{x^2}\Big\}a_1$$
$$=0\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where

$$f=\frac{9n^2(p+1)}{4\pi G\bar{\rho}p\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Assume the following series solution for (38):

$$a_1 = x^q \sum_0^\infty C_\lambda x^\lambda . \qquad (40)$$

The indicial equation gives

$$q = \frac{(p-2) \pm \sqrt{(p-2)^2 + 4(p+1)\alpha}}{2} \qquad (41)$$

As γ lies[8] between $\frac{4}{3}$ and $\frac{5}{3}$, α given by (7) lies between 0 and ·6, and we choose only the positive root of (41) to avoid divergence at the origin (centre).

We have the recurrence formula

$$C_{\lambda+p+1}[(\lambda+q+p+1)(\lambda+q+3)-\alpha(p+1)]$$
$$=(\lambda+q)(\lambda+q+3)C_\lambda - fC_{\lambda+p-2} \qquad (42)$$

Dividing (42) by λ^2 and proceeding to the limit as $\lambda \to \infty$, we have

$$1 = \mathrm{Lt}\frac{C_{\lambda+p+1}}{C_\lambda} = \left(\mathrm{Lt}\frac{C_{\lambda+1}}{C_\lambda}\right)^{p+1}. \qquad (43)$$

whence we have

$$\mathrm{Lt}\frac{C_{\lambda+1}}{C_\lambda} = 1. \qquad (44)$$

Hence, the series solution for (38) has unit radius of convergence. This is also evident[12] from the fact that the differential equation (38) has regular singularities at $x=0$ and $x=1$.

From (42), we have

$$f\frac{C_{\lambda+p-2}}{C_\lambda} = \lambda^2\left(1-\frac{C_{\lambda+p+1}}{C_\lambda}\right) + \lambda\left\{(2q+3)-(2q+p+4)\frac{C_{\lambda+p+1}}{C_\lambda}\right\}$$
$$+q(q+3) - \left\{(q+p+1)(q+3)-\alpha(p+1)\right\}\frac{C_{\lambda+p+1}}{C_\lambda}. \qquad (45)$$

From (45), we can show as in (36) that

$$\frac{C_{\lambda+1}}{C_\lambda} = 1 - \frac{1}{\lambda} + 0\left(\frac{1}{\lambda^2}\right)^{23} \qquad (46)$$

(46) shows that the series solution (40) is divergent for $x=1$. (Gauss's rule[2])

Instability of oscillations for the model follows from the extension of Abel's theorem.[3]

The present analysis leads to the following conclusion:

If the density vary inversely as the pth power of the distance from the centre, where $p=0$ or a positive integer excluding 1 and 3, no mode of radial oscillation is possible except for Sterne's[18] models (1) the sphere of uniform density and

(2) the sphere of vanishingly small core in which the density varies inversely as the square of the distance from the centre.

Our conclusion has an important bearing on the constitution of the Cepheid Variables which are known to be large diffuse gaseous masses of very little central condensation,[20] that is, approximately homogeneous; and it has been followed up in two other papers by the author[20, 21].

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ON THE THEORY OF A SPIRAL NEBULA

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The shapes of spiral nabulæ have raised great difficulties, and several theories have been put forward to explain their formation.

Jeans[5] has worked out the case of a rotating mass of matter, consisting of a point nucleus and surrounded by an atmosphere of negligible total mass. He obtained a series of configurations with increasing rates of rotation. He supposes that when the rate of rotation has sufficiently increased, matter would be thrown out in the equatorial plane along spiral arms. His theory rules out the possibility of star-clouds existing in the ellipsoidal nebulæ, which is assumed to be gaseous in nature. This is not corroborated by observation.

Brown[4] assumes that originally every spiral nebula was a highly flattened homogeneous ellipsoid of revolution inside which the gravitational force of attraction is of the form $-AX$, $-AY$, $-CZ$. Later on minor changes in the uniform density are assumed to be due to perturbations caused by rather close encounters with passing galaxies. These perturbations ultimately lead to the formation of the spiral arms of the nebula. He further concludes that after the encounters the spiral arms gradually coil up and the nebula ultimately reverts to its original shape. According to him the spiral form is not a permanent one and its formation is being repeated more or less periodically by encounters. There is a great drawback in his theory—apart from the small superposition he assumes uniform density throughout the galaxy which is not borne out by observation. Moreover, there is no evidence to show that spiral formation is a periodic phenomenon.

In Vogt's[9] theory most of the mass of the spiral nebula is supposed to be concentrated in the nucleus, so that everywhere outside the nucleus the force of attraction may be taken to vary as $\frac{1}{r^2}$. In addition to this, he assumes that there is a force of repulsion proportional to the distance from the centre of the nucleus. The main objection to his theory is his assumption of the repulsive force.

Lindblad[6] assumed in his earlier theory that there was small condensed nucleus which is surrounded by a spheroidal galaxy of stars of uniform density from which spiral arms emanate. But in a recent paper, he states that "The stellar system may be divided into a number of sub-systems of approximately the same

extension in the galactic plane, but with different degree of flattening towards the plane and different speed of rotation at the same distance from the axis."

More recently Banerji,[1] Bhatnagar and Nizamuddin have investigated the condition necessary for the formation of the spiral arms on the basis of a rotating spheroidal central mass of finite dimensions and of uniform density surrounded by a spheroidal structure of rotating compressible gas of variable density. In a later paper, on the basis of Eddington's theoretical researches and on the observed feature of galactic rotation which may be ascribed to a highly concentrated central mass together with a uniform spheroidal distribution of matter, Banerji[2] has investigated the condition necessary for the formation of spiral arms in the equatorial plane of a rotating gaseous configuration of uniform density which surrounds a spheroidal homogeneous mass of incompressible material.

In this paper we have also assumed Banerji's configuration of a rotating speroidal central mass of finite dimensions and of uniform density surrounded by a spheroidal structure of rotating compressible gas of variable density. We have investigated the actual path of an ejected material from the equatorial plane and the condition necessary for the formation of the spiral nebula when resistance is introduced. It may be mentioned here that as intergalactic space is not totally free from matter so the ejected matter would experience some resistance, however small it may be, as it passes through the resisting medium. Moreover we have also shown the possibility for the formation of irregular nebulæ.

The force of attraction measured positively towards the centre is taken to be f[3].

$$f=\tfrac{4}{3}\pi a^3\rho_0 k_0{}^2(1-e^2)^{\frac{1}{2}}\left[\frac{3-2k_0}{r^2}+\frac{a^2e^2}{r^4}\cdot\frac{(5-2k_0{}^3)}{10}\right]+\cdots\cdots\cdots$$

$$+\frac{a^{2n}e^{2n}}{r^{2n+2}}\cdot\frac{1\,.\,3\ldots 2n-1}{2\,.\,4\ldots\ldots 2n}\cdot\frac{3(2n+3-2k_0{}^{2n+1})}{(2n+1)(2n+3)}\Bigg]$$

By keeping terms only up to e^{2n}

$$f=\frac{A_0}{r^2}+\frac{A_1}{r^4}+\cdots\cdots\cdots\cdots\frac{A_n}{r^{2n+2}}$$

where $A_0=\tfrac{4}{3}\pi a^3\rho_0 k_0{}^2(1-e^2)^{\frac{1}{2}}(3-2k_0)=\tfrac{4}{3}\pi a^3\rho_0 k_0{}^2(1-e^2)^{\frac{1}{2}}\,\alpha_0$

. . . .

$$A_n=\tfrac{4}{3}\pi a^3\rho_0 k_0{}^2(1-e^2)^{\frac{1}{2}}\;a^{2n}\,e^{2n}.\;\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n}\cdot 3\left\{\frac{(2n+3)-2k_0{}^{2n+1}}{(2n+1)(2n+3)}\right\}$$

$$=\tfrac{4}{3}\pi a^3\rho_0 k_0{}^2(1-e^2)^{\frac{1}{2}}\;a^{2n}.\,e^{2n}.\,\alpha_n$$

where

$$\alpha_0 = 3 - 2k_0$$

$$\frac{\alpha_1 = 5 - 2k_0{}^3}{10}$$

. . . .

$$\alpha n = \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots \ldots 2n} \cdot 3 \cdot \frac{(2n+3-2k_0{}^{2n+1})}{(2n+1)(2n+3)}.$$

The equation of the orbit of an external particle which has been just ejected the equatorial plane is

$$\frac{d2u}{d\theta^2} + u = \frac{f}{h^2 u^2} = \frac{1}{h^2}\left[A_0 + A_1 u^2 + \ldots . A_n u^{2n}\right]$$

where $u = \frac{1}{r}$ and $h = a^2 w$.

1. The condition for the ejection of matter by the rotating mass and formation of irregular nebula.

Consider the forces acting on a particle of rotating mass lying in the equator plane. A centrifugal force $\omega^2 r$ is acting on the particle tending an outward moti and secondly a force of attraction f is acting on the particle. Now

$$f = \frac{A_0}{r^2} + \frac{A_1}{r^4} + \ldots \ldots \frac{A_n}{r^{2n+2}}.$$

The condition for the ejection can be written

$$\omega^2 a > \frac{A_0}{a^2} + \frac{A_1}{a^4} + \ldots \ldots \frac{A_n}{a^{2n+2}}$$

or

$$\omega^2 a^3 > A_0 + \frac{A_1}{a^2} + \ldots \ldots \frac{A_n}{a^{2n}}.$$

If the original points of ejection are distributed at random many in number quite irregular forms of arms is expected which may give rise to irregular sha nebula.

But if the ejection of matter be confined from the two diametrically opposi points [7] on the circumference of the central system, spiral arms are formed.

2. The equation of the path of an external particle which has been ju ejected in the equatorial plane is

$$\frac{d2u}{d\theta^2} + u = \frac{1}{h^2}\left[A_0 + A_1 u^2 + \ldots \ldots A_n u^{2n}\right]$$

The effect of some such disturbing factor as tidal action will be to cause a slight perturbation, but this will leave h unchanged, *i.e.*, $h=a^2\omega$, ω being the angular velocity. Therefore,

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2}{h^2}\left[\ A_0u+\frac{A_1}{3}u^3+\ \ldots\ \frac{A_n}{(2n+1)}u^{2n+1}\ \right]+E$$

where E is the constant of integration. We can get the value of E by putting the initial condition, *i.e.*, $u=u_0=\frac{1}{a}$ and $\frac{du}{d\theta}=0$. We get

$$\left(\frac{du}{d\theta}\right)^2=\frac{2}{h^2}\left[\ A_0(u-u_0)+\frac{A_1}{3}(u^3-u_0^3)+\ldots+\frac{A_n}{(2n+1)}\left(\ u^{2n+1}-u_0^{2n+1}\right)\right]$$
$$+u_0^2-u^2.$$

Putting the value of the constant

$$2\cdot\frac{4}{3}\ \frac{\pi\rho_0k_0^2\alpha_0}{\omega^2}\ \cdot\frac{1}{a}=mu_0$$

where m is positive and small and less than I.

On substituting the values of ρ_0, k_0^2, α_0 and ω^2 we see that m is actually less than I, i.e., $m<1$.

$$\left(\frac{du}{d\theta}\right)^2=mu_0\,(1-e^2)^{\frac{1}{2}}\left[(u-u_0)+\frac{\alpha_1e^2}{3\alpha_0u_0^2}(u^3-u_0^3)\right.$$
$$\left.+\ \frac{\alpha_2e^4}{5\alpha_0u_0^4}(u^5-u_0^5)\right]+u_0^2-u^2.$$

By keeping only up to e^4, we have

$$\left(\frac{du}{d\theta}\right)^2=mu_0\left[(u-u_0)+\frac{e^2}{3\alpha_0}\cdot\frac{\alpha_1}{u_0^2}(u^3-u_0^3)-\frac{e^2}{2}\ (u-u_0)\right.$$
$$\left.+e^4\left\{\frac{\alpha_2(u^5-u_0^5)}{5\alpha_0u_0^4}-\frac{\alpha_1}{3\alpha_0u_0^2}\cdot\frac{(u^3-u_0^3)}{2}-\tfrac{1}{8}\,(u-u_0)\right\}\right]$$
$$+u_0^2-u^2$$

$$\frac{du}{d\theta}=\pm\,[(u_0-u)\{u+u_0(1-m)\}]^{\frac{1}{2}}\left[\ 1+\frac{e^2mu_0\left\{\frac{1}{2}-\frac{\alpha_1}{3\alpha_0}\cdot\frac{(u^2+uu_0+u_0^2)}{u_0^2}\right\}}{u+u_0\,(1-m)}\right.$$
$$\left.+e^4\frac{\left\{\frac{1}{8}+\frac{\alpha_1(u^2+uu_0+u_0^2)}{2\cdot3\cdot\alpha_0u_0^2}-\frac{\alpha_2}{5\alpha_0u_0^4}(u^4+u^3u_0+u^2u_0^2+uu_0^3+u_0^4)\right\}}{u+u_0\,(1-m)}\right]^{\frac{1}{2}}$$

or,

$$\int \frac{du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0(1-m)]^{\frac{1}{2}}} + \tfrac{1}{2}e^2 m u_0 \int \frac{\left[\dfrac{\alpha_1(u^2+uu_0+u_0^2)}{3\alpha_0 u_0^2} - \tfrac{1}{2}\right] du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0\,(1-m)]^{\frac{3}{2}}}$$

$$+\tfrac{1}{2}\,e^2 m u_0 \int \frac{\left\{\dfrac{\alpha_2}{5\alpha_0 u_0^4}(u^4+u^3u_0+u^2u_0^2+uu_0^3+u_0^4)-\dfrac{1}{8}-\dfrac{\alpha_1}{2\cdot 3\alpha_0 u_0^2}\right.}{(u_0-u)^{\frac{1}{2}}\,[u+u_0(1-m)^{\frac{3}{2}}}$$

$$\left.(u^2+uu_0+u_0^2)\right\} du$$

$$+\tfrac{3}{8}\, m^2 e^4 u_0^2 \int \frac{\left\{\tfrac{1}{2}-\dfrac{\alpha_1}{3\alpha_0 u_0^2}(u^2+uu_0+u_0^2\right\}^2 du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0\,(1-m)]^{\frac{5}{2}}} = \theta + c.$$

First we consider the path, neglecting e^2 and its higher powers.

We have
$$\int \frac{du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0\,(1-m)]^{\frac{1}{2}}} = \theta + c$$

$$2\sin^{-1}\sqrt{\frac{u+u_0\,(1-m)}{(2-m)\,u_0}} = \theta + c$$

We can find the value of the constant of integration c by noticing that when $u=u_0$, $\theta=0$ we have $c=\pi$.

Then, we have

$$\frac{2u}{(2-m)\,u_0} = \frac{m}{(2-m)} + \cos\theta$$

Here two cases arise (i) if $m=1$ then the path is a parabola, (ii) if $m<1$ then the path is a hyperbola. Secondly, we consider the terms only up to e^2.

$$\int \frac{du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0\,(1-m)]^{\frac{1}{2}}} + \tfrac{1}{2} m u_0 e^2 \int \frac{\left\{\dfrac{\alpha_1}{3\alpha_0 u_0^2}(u^2+uu_0+u_0^2)-\tfrac{1}{2}\right\} du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0\,(1-m)]^{\frac{1}{2}}}$$

$$=\theta + c_1$$

$$\left[1+\frac{\alpha_1 m^2 e^2}{4\alpha_0}\right]\cdot 2\sin^{-1}\sqrt{\frac{u+u_0\,(1-m)}{(2-m)\,u_0}} - e^2 m u_0$$

$$\left[\frac{\alpha_1}{6\alpha_0 u_0^2}\cdot\sqrt{(u_0-u)\,[u+u_0(1-m)]} + \frac{\{\alpha_1/3\,\alpha_0{}^{(1+m^2-m)}-\tfrac{1}{2}\}}{(2-m\,u_0}\sqrt{\frac{u_0-u}{(u+u_0(1-m)}}\right]$$

$$=\theta + c_1$$

The value of the constant of integration c_1 can be found out as before by putting $u=u_0$, $\theta=0$.

Thus, we have

$$\sin\left[\frac{\theta}{2\left(1+\frac{\alpha_1 m^2 e^2}{4\alpha_0}\right)}+\frac{\pi}{2}+e^2 m u_0\left\{\frac{\alpha_1}{6\alpha_0 u_0^2}\cdot\sqrt{\frac{(u_0-u)\{u+u_0(1-m)}{1}}\right.\right.$$

$$\left.\left.+\frac{B_1}{u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right\}\right]$$

$$=\sqrt{\frac{u+u_0(1-m)}{u_0(2-m)}}$$

where

$$B_1=\frac{\frac{\alpha_1}{3\alpha_0}(1+m^2-m)-\frac{1}{2}}{(2-m)}.$$

In left-hand side of the above put

$$\frac{2u}{(2-m)u_0}=\frac{m}{(2-m)}+\cos\theta$$

and keeping the terms only up to e^2, we have

$$\frac{2u}{mu_0}=1+\frac{2-m}{m}\cdot\cos\left[\theta\left(1-\frac{\alpha_1 e^2 m^2}{4\alpha_0}\right)+e^2 m\left\{\frac{\alpha_1}{6\alpha_0}\left(1-\frac{m}{2}\right)\sin\theta+\beta_1\right.\right.$$

$$\left.\left.\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}\right]$$

Now put $\quad \frac{\alpha_1 e^2 m^2}{4\alpha_0}=\varepsilon,\ e^2 m\left[\frac{\alpha_1}{6\alpha_0}\left(1-\frac{m}{2}\right)\right]=\gamma$ and $e^2 m\beta_1=\delta$

where ε and γ are positive and δ is $-$ve.

On expanding and taking proper approximation, we have

$$\frac{2u}{mu_0}=1+\frac{2-m}{m}\left[\cos\theta+\left\{\varepsilon\,\theta-\gamma\sin\theta-\delta\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}\sin\theta\right]$$

or,

$$\frac{2u}{mu_0}=1+\frac{2-m}{m}\left[\cos\theta+P\sin\theta\right]$$

where $P=\varepsilon\,\theta-\gamma\sin\theta-\delta\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$, which is always positive and $0\,(e^2)$.

Now again we take two cases (i) if $m=1$ we see that above path is a deformed parabola and the factor $P\sin\theta$ makes the path more bent towards the axis, *i.e.*, more spiral like form (ii) if $m<1$ we have a deformed hyperbola and again the deforming factor $P\sin\theta$ makes the path more bent towards the axis.

F. 5

Therefore by considering terms up to e^2, we see that the particle bends more towards the axis than without the terms containing e^2. We expect that the particle will bend still more towards the axis if we consider the terms containing higher powers of e^2.

Next we consider the terms up to e^4, we have

$$\int \frac{du}{(u_0-u)^{\frac{1}{2}}[u+u_0(1-m)]^{\frac{1}{2}}}+\tfrac{1}{2}e^2 m u_0 \int \frac{\left[\frac{\alpha_1}{3\alpha_0 u_0^2}(u^2+uu_0+u_0^2)-\frac{1}{2}\right]du}{[u_0-u]^{\frac{1}{2}}\,[u+u_0(1-m)]^{\frac{3}{2}}}$$

$$+\tfrac{1}{2}e^4 m u_0 \int \frac{\left[\frac{\alpha_2}{5\alpha_0 u_0^4}(u^4+u^3u_0+u^2u_0^2+uu_0^3+u_0^4)-\frac{1}{8}-\frac{\alpha_1}{6\alpha_0 u_0^2}(u^2+uu_0+u_0^2)\right]du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0(1-m)]^{\frac{3}{2}}}$$

$$+\tfrac{3}{8}m^2e^4u_0^2\int \frac{\left[\frac{1}{2}-\frac{\alpha_1}{3\alpha_0u_0^2}(u^2+uu_0+u_0^2)\right]^2du}{(u_0-u)^{\frac{1}{2}}\,[u+u_0(1-m)]^{\frac{5}{2}}}=\theta+c$$

which gives on integration

$$\left[1+e^2\beta_0+e^4(\beta_2-\beta_0+\beta_3)\right]2\sin^{-1}\sqrt{\frac{u+u_0(1-m)}{(2-m)u_0}}$$

$$+e^2mu_0\left[\frac{\alpha_1}{6\alpha_0u_0^2}\left\{-\sqrt{(u_0-u)[u+u_0(1-m)]}\right\}-\frac{\beta_1}{u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right]$$

$$+\frac{e^4\alpha_2 m u_0}{10\alpha_0u_0^4}\left[\left\{-\frac{[u+u_0(1-m)]^2}{3}+c_0u_0[u+u_0(1-m)]+c_1\right\}\sqrt{(u_0-u)\,[u+u_0(1-m)]}\right.$$

$$\left.-2e^4mc_2\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right]+\frac{e^4m^2\alpha_1}{12\alpha_0u_0}\left[\sqrt{u_0-u\,[u+u_0(1-m)]}+2c_3\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right]$$

$$-\frac{e^4\alpha_1^2m^2}{24\alpha_0^2u_0^2}\left\{\frac{(11mu_0-2u)}{4}\sqrt{(u_0-u)[u+u_0(1-m)]}+2(2m-1)u_0^2c_3\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right\}$$

$$-2e^4m^2u_0c_4\frac{(u_0-u)^{\frac{1}{2}}}{[u+u_0(1-m)]^{\frac{3}{2}}}\left\{1-\frac{2}{3}\,\frac{(u_0-u)}{(2-m)u_0}\right\}$$

$$+\frac{e^4m^2\alpha_1}{8\alpha_0}\left[\frac{2(2m-1)u_0}{(2-m)u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}+2u_0c_3\,\frac{[u_0-u]^{\frac{1}{2}}}{[u+u_0(1-m)]^{\frac{3}{2}}}\left\{1-\frac{2}{3}\,\frac{u_0-u}{(2-m)u_0}\right\}\right]$$

$=\theta+$ constant

$$\text{where}\quad \beta_0=\frac{\alpha_1 m^2}{4\alpha_0}.\quad \frac{\frac{\alpha_1}{3\alpha_0}(1+m^2-m)-\frac{1}{2}}{(2-m)}=B_1$$

$$\beta_2=\frac{m\alpha_2}{10\alpha_1}.\quad \frac{35m^3-60m^2+60m}{16}$$

$$\beta_3=\frac{\alpha_1{}^2 m^2}{24\alpha_0{}^2}.\quad \frac{35m^2-20m+20}{8}$$

$$C_0=\frac{8-19m}{12},\quad C_1=\frac{26m-29m^2-16}{8}$$

$$C_2=\frac{\frac{\alpha_2}{10\alpha_0}(m^4-3m^3+4m^2-2m)-\frac{1}{16}}{(2-m)}$$

$$C_3=\frac{1+m^2-m}{2-m}\quad\text{and}\quad C_4=\frac{\frac{\alpha_1{}^2}{24\alpha_0}(1+m^2-m)^2+\frac{3}{32}}{(2-m)}$$

The constant of integration can be determined by putting $u=u_0$, $\theta=0$, we have,

constant$=\pi\{1+e^2\beta_0+e^4\ (\beta_2-\beta_0+\beta_3)\}$.

$$2\sin^{-1}\sqrt{\frac{u+u_0(1-m)}{u_0(2-m)}}=[\theta\{1+e^2\beta_0+e^4(\beta_2-\beta_0+\beta_3)\}+\pi$$

$$+\{1+e^2\beta_0+e^4\ (\beta_2-\beta_0+\beta_3)\}^{-1}\Bigg[e^2 m u_0\Bigg\{\frac{\alpha_1}{6\alpha_0 u_0{}^2}\sqrt{(u_0-u)[u+u_0(1-m)]}$$

$$+\frac{\beta_1}{u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\Bigg\}-\frac{e^4\alpha_2 m u_0}{10\alpha_0 u_0{}^4}\Bigg\{\Bigg(-\frac{[u+u_0(1-m)]^2}{3}+c_0 u_0[u+u_0(1-m)]+c_1\Bigg)$$

$$\sqrt{(u_0-u)[u+u_0(1-m)]}\Bigg\}+2e^4 m c_2\sqrt{\frac{(u_0-u)}{u+u_0(1-m)}}$$

$$-\frac{e^4 m\alpha_1}{12\alpha_0 u_0}\Bigg\{\sqrt{(u_0-u)[u+u_0(1-m)]}+2c_3\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\Bigg\}$$

$$+\frac{e^4\alpha_1{}^2 m^2}{24\alpha_0{}^2 u_0{}^2}\Bigg\{\Bigg(\frac{11mu_0-2u}{4}\Bigg)\sqrt{(u_0-u)[u+u_0(1-m)]}+2(2m-1)u_0{}^2 c_3\sqrt{\frac{(u_0-u)}{u+u_0(1-m)}}\Bigg\}$$

$$+2e^4 m^2 u_0 c_4.\frac{[u_0-u]^{\frac{1}{2}}}{[u+u_0(1-m)]^{\frac{3}{2}}}\Bigg\{1-\frac{2}{3}.\frac{(u_0-u)}{(2-m)u_0}\Bigg\}$$

$$+\frac{e^4 m^2\alpha_1}{8\alpha_0}\Bigg\{\frac{2(2m-1)u_0}{(2-m)u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}+2u_0 c_3\frac{[u_0-u]^{\frac{1}{2}}}{[u+u_0(1-m)]^{\frac{3}{2}}}\Bigg\{1-\frac{2}{3}\frac{u_0-u}{(2-m)u_0}\Bigg\}\Bigg\}$$

Now keeping the terms only up to e^4, we have

$$2\sin^{-1}\sqrt{\frac{u+u_0(1-m)}{(2-m)u_0}} = \left\{ \theta\left[1-e^2\beta_0-e^4(\beta_2-\beta_0-\beta_0^2+\beta_3)\right]+\pi\right.$$

$$+e^2 m u_0\left[\frac{\alpha_1}{6\alpha_0 u_0^2}\sqrt{(u_0-u)[u+u_0(1-m)]}+\frac{\beta_1}{u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right]$$

$$-\frac{e^4\alpha_2 m u_0}{10\alpha_0 u_0^4}\left[\left\{-\frac{[u+u_0(1-m)]^2}{3}+c_0u_0[u+u_0(1-m)]+c_1\right\}\right.$$

$$\left.\sqrt{(u_0-u)[u+u_0(1-m)]}\right]$$

$$+2e^4mc_2\sqrt{\frac{u_0-u}{u+u_0(1-m)}}-\frac{e^4 m\alpha_1}{12\alpha_0 u_0}\left[\sqrt{(u_0-u)[u+u_0(1-m)]}+2c_3\right.$$

$$\left.\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right]$$

$$+\frac{e^4\alpha_1^2m^2}{24\alpha_0^2u_0^2}\left\{\frac{(11mu_0-2u)}{4}\sqrt{(u_0-u)[u+u_0(1-m)]}+2(2m-1)u_0^2c_3\right.$$

$$\left.\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right\}$$

$$+2e^4m^2u_0c_4\frac{[u_0-u]^{\frac{1}{2}}}{[u+u_0(1-m)]^{\frac{3}{2}}}\left\{1-\frac{2}{3}\frac{(u_0-u)}{(2-m)u_0}\right\}$$

$$+\frac{e^4m^2\alpha_1}{8\alpha_0}\left\{\left[\frac{2(2m-1)}{(2-m)}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}+2u_0c_3\frac{[u_0-u]^{\frac{1}{2}}}{[u+u_0(1-m)]^{\frac{3}{2}}}\right.\right.$$

$$\left.\left.\left\{1-\frac{2}{3}\frac{(u_0-u)}{(2-m)u_0}\right\}\right]\right.$$

$$\left.-e^4\beta_0 m u_0\left[\frac{\alpha_1}{6\alpha_0 u_0^2}\cdot\frac{\sqrt{(u_0-u)(u+u_0(1-m))}}{1}+\frac{\beta_1}{u_0}\sqrt{\frac{u_0-u}{u+u_0(1-m)}}\right]\right\}$$

By substituting $\dfrac{2u}{mu_0}=1+\dfrac{2-m}{m}\Big[\cos\theta+\mathrm{P}\sin\theta\Big]$

or $u=u_0\left\{\dfrac{m}{2}+\dfrac{2-m}{2}(\cos\theta+\mathrm{P}\sin\theta)\right\}$ in the above equation and keeping terms only up to e^4, we have

$$\frac{2u}{mu_0}=1+\frac{2-m}{m}\cdot\cos\Big[\theta\left\{1-e^2\beta_0-e^4(\beta_2-\beta_0-\beta_0^2+\beta_3)\right\}$$

$$+e^2m\left\{\frac{\alpha_1}{6\alpha_0}\left(1-\frac{m}{2}\right)(\sin\theta-\mathrm{P}\cos\theta)+\beta_1\left(1-\frac{\mathrm{P}}{\sin\theta}\right)\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$+e^4\left[\frac{\alpha_2 m}{10\alpha_0}\left\{\frac{\left(1-\frac{m}{2}\right)^2(1+\cos\theta)^2}{3}-c_0\left(1-\frac{m}{2}\right)(1+\cos\theta)+c_1\right\}\right.$$

$$\left(1-\frac{m}{2}\right)\sin\theta$$

$$+2mc_2\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}-\frac{m\alpha_1}{12\alpha_0}\left[\left(1-\frac{m}{2}\right)\sin\theta+2c_3\cdot\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

$$+\frac{\alpha_1{}^2m^2}{24\alpha_0{}^2}\left[\frac{(10m-(2-m)\cos\theta)}{4}.\left(1-\frac{m}{2}\right)\sin\theta+2(2m-1)c_3\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

$$+2m^2c_4\frac{[1-\cos\theta]^{\frac{1}{2}}}{[1+\cos\theta]^{\frac{3}{2}}}.\frac{2}{3}.\frac{(2+\cos\theta)}{(2-m)}$$

$$+\frac{\alpha_1 m^2}{8\alpha_0}\left[\frac{2(2m-1)}{(2-m)}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}+2c_3\frac{[1-\cos\theta]^{\frac{1}{2}}}{[1+\cos\theta]^{\frac{3}{2}}}\;\frac{2+\cos\theta}{(2-m)}\cdot\frac{2}{3}\right.$$

$$-\beta_0 m\left[\frac{\alpha_1}{6\alpha_0}\left(1-\frac{m}{2}\right)\sin\theta+\beta_1\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

or

$$\frac{2u}{u_0m}=1+\frac{2-m}{m}\cos\left[\theta\left\{1-e^2\beta_0-e^4(\beta_2-\beta_0-\beta_0{}^2+\beta_3)\right\}\right.$$

$$+e^2m\left\{\frac{\alpha_1}{6\alpha_0}\left(1-\frac{m}{2}\right)(\sin\theta-P\cos\theta)+\beta_1\left(1-\frac{P}{\sin\theta}\right)\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$+e^4\left\{\left\{\frac{\alpha_2 m}{10\alpha_0}\left[\frac{\left(1-\frac{m}{2}\right)^2(1+\cos\theta)^2}{3}-c_0\left(1-\frac{m}{2}\right)(1+\cos\theta)+c_1\right]\left(1-\frac{m}{2}\right)\sin\theta\right\}\right.$$

$$+K\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}+\left(1-\frac{m}{2}\right)\sin\theta\left\{\frac{\alpha_1{}^2m^2}{24\alpha_0{}^2}.\frac{10m-(2-m)\cos\theta}{4}-\right.$$

$$\left.\frac{m\sigma_1}{12\alpha_0}-\frac{\beta_0 m\alpha_1}{6\alpha_0}\right\}+\left[\frac{2}{3}.\frac{(2+\cos\theta)}{(2-m)}\frac{[1-\cos\theta]^{\frac{1}{2}}}{[1+\cos\theta]^{\frac{3}{2}}}\right]\times\left[2m^2c_4+\frac{\alpha_1 m^2 2c_3}{8\alpha_0}\right]\Bigg\}$$

$$\text{where } K=\left[2mc_2-\frac{m\alpha_1}{12\alpha_0}.2c_3+2(2m-1)c_3.\frac{\alpha_1{}^2m^2}{24\alpha_0{}^2}+\frac{\alpha_1 m^2}{8\alpha_0}.\frac{2(2m-1)}{(2-m)}-\beta_0\beta_1 m\right]$$

which gives the path

3. Now space outside a nebula is not totally devoid of matter, and thus the ejected particle would therefore travel in a resisting medium whose density decreases as the distance increases from the centre. It will not be out of place to assume

that the particle would experience a tangential resistance per unit mass. of amount $\frac{Lv}{r^2}$, where v is the velocity of the particle, r its distance from the centre of the nebula and the constant L is very small.

The transverse and radial components are

$$-\frac{L}{r^2}r\dot{\theta} \text{ and } -\frac{L}{r^2}\cdot\frac{dr}{dt}$$

Equations of motion now become

$$\ddot{r}-r\dot{\theta}^2=-\frac{1}{r^2}\left(A_0+\frac{A_1}{r^2}+\ldots\ldots\frac{A_n}{r^{2n}}\right)-\frac{L}{r^2}\cdot\dot{r} \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=-\frac{L}{r^2}r\dot{\theta} \quad . \quad . \quad . \quad (2)$$

where $A_0, A_1 \ldots\ldots A_n$ are given in previous section.

Now equation (1) can be written as

$$\frac{d}{dt}(-r^2\dot{r})+2r\dot{r}^2+r^3\dot{\theta}^2-L\dot{r}=\left(A_0+\frac{A_1}{r^2}+\ldots\ldots\frac{A_n}{r^{2n}}\right)$$

Put $r=\frac{1}{u}$, $\dot{r}=-\frac{1}{u^2}\dot{u}$

$$\frac{d}{dt}(u^{-4}\dot{u})+2u^{-5}\dot{u}^2+u^{-3}\dot{\theta}^2=(A_0+\ldots\ldots A_n u^{2n})-Lu^{-2}\dot{u}$$

Now equation (2) becomes

$$\frac{d}{dt}(u^{-2}\dot{\theta})=-L\dot{\theta}$$

or $\quad u^{-2}\dot{\theta}=h_0-L\theta \qquad$ (by integration)

Let $\quad u^{-2}\dot{\theta}=H=h_0-L\theta$

or $\quad \frac{d}{dt}=Hu^2\frac{d}{d\theta}$

Now the above equation becomes

$$H^2\left[\frac{d_2u}{d\theta^2}+u\right]+H\left[\frac{dH}{d\theta}+L\right]\frac{du}{d\theta}=A_0+A_1u^2+\ldots\ldots A_nu^{2n}$$

By putting the value of $\frac{dH}{d\theta}$, the above reduces to

$$H^2\left[\frac{d_2u}{d\theta^2}+u\right]=A_0+A_1u^2+\ldots\ldots+A_nu^{2n}$$

or

$$\frac{d_2u}{d\theta^2}+u=\left[\frac{A_0+A_1u^2+\ldots\ldots+A_nu^{2n}}{h_0^2}\right]\left[1+\frac{2L\theta}{h_0}\right]$$

where higher powers of L are neglected :

By integrating it, we get

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2}{h_0^2}\left[A_0u+\ldots\ldots+\frac{A_n}{2n+1}u^{2n+1}\right]+\frac{4L}{h_0^3}\int(A_0+\ldots\ldots A_nu^{2n})\theta du+E$$

where E is the constant of integration

$$\left(\frac{du}{d\theta}\right)^2=\frac{2}{h_0^2}\left(A_0u+\frac{A_1u^3}{3}+\ldots\ldots+\frac{A_n}{(2n+1)}u^{2n+1}\right)-u^2$$
$$+\frac{4L}{h_0^3}\int\left(A_0+A_1u^2+\ldots\ldots A_nu^{2n}\right)\theta du+E$$

We have, therefore,

$$\frac{dr}{d\theta}=\pm r^2\sqrt{\frac{2}{h_0^2}\left((A_0u+\ldots\ldots\frac{A_n}{2n+1}u^{2n+1}\right)-\frac{2}{h_0^2}\left(A_0u_0+\ldots\ldots\frac{A_n}{(2n+1)}u_0^{2n+1}\right)+\frac{1}{a^2}-\frac{1}{r^2}+\frac{4L}{h_0^3}\int(A_0+\ldots\ldots A_nu^{2n})\theta du+c}$$

Now for a spiral form, $\frac{dr}{d\theta}$ must be real, finite, continuous and of the same sign as r changes with θ. It is, therefore, necessary that the expression under the root sign must be positive for all values of $r>a$.

$$\frac{r^2-a^2}{r^2a^2}+\frac{4L}{h_0^3}\int(A_0+A_1u+\ldots\ldots+A_nu^{2n})\theta du+c_1>$$
$$\frac{2}{h_0^2}\cdot\tfrac{4}{3}\pi a^3(1-e^2)^{\frac{1}{2}}\rho_0\,k_0^2\left[\frac{r-a}{ra}\right]\left[\alpha_0+\frac{\alpha_1e^2}{3}\left(1+\frac{a}{r}+\frac{a^2}{r^2}\right)+\ldots\ldots\right.$$
$$\left.+\frac{\alpha_ne^{2n}}{(2n+1)}\left(1+\frac{a}{r}+\ldots\ldots+\frac{a^{2n}}{r^{2n}}\right)\right]$$

Now if $L=0$, *i.e.*, particle moves free, then the above condition is reduced to the same as obtained by Banerji and Bhatnagar. Now for a particular case when e is very small and $\theta=\sin^{-1}\left(\frac{2au-m}{2-m}\right)-\frac{\pi}{2}$ taken from the previous section.

$$\frac{d2u}{d\theta2}+u=\frac{1}{h_0^2}\pi\cdot\tfrac{4}{3}a^3\rho_0k_0^2\alpha_0+\frac{2L}{h_0^3}\left[\sin^{-1}\frac{2au-m}{2-m}-\frac{\pi}{2}\right]\alpha\tfrac{4}{3}\pi a^3\rho_0k_0^2$$

Neglecting e, we have

$$\left(\frac{du}{d\theta}\right)^2 = mu_0 u + \frac{2L}{ho} \cdot mu_0 \left\{ \left[u \sin^{-1}\left(\frac{2au-m}{2-m}\right) - \frac{\pi}{2} u \right] + \frac{u_0}{2} \left[(2-m)^2 - (2au-m)^2 \right]^{\frac{1}{2}} - \frac{m}{2a} \sin^{-1}\left(\frac{2au-m}{2-m}\right) \right\} + E - u^2$$

$$\frac{dr}{d\theta} = \pm r^2 \sqrt{ mu_0 (u-u_0) + \frac{2L\, mu_0}{ho} \left[u_0 (1-au)^{\frac{1}{2}} (1+au-m)^{\frac{1}{2}} + \left(\frac{\pi}{2} - \sin^{-1}\frac{2au-m}{2-m}\right)\left(\frac{m}{2a} - u\right) \right] + u_0{}^2 - u^2 }$$

Thus the condition is reduced to

$$\frac{1}{a^2} - \frac{1}{r^2} + 2\frac{Lm}{h_0 a}\left[\frac{1}{a}\left(\left(1-\frac{a}{r}\right)^{\frac{1}{2}}\left(1+\frac{a}{r}-m\right)^{\frac{1}{2}} + \left\{\frac{\pi}{2} - \sin^{-1}\left(\frac{\frac{2a}{r}-m}{2-m}\right)\right\} \times \left\{\left(\frac{m}{2a} - \frac{1}{r}\right)\right\}\right] > \frac{m}{a}\left(\frac{1}{a} - \frac{1}{r}\right).$$

In the end I think it my great privilege to record my grateful thanks to Prof. A. C. Banerji under whose guidance the present paper has been produced.

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MOTION OF AN INCOMPRESSIBLE FLUID WITH VARYING COEFFICIENT OF VISCOSITY, GIVEN BY $\mu=\mu_0+\varepsilon_1 x$, FOR POSITIVE VALUES OF x. PART I

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Various investigators have considered the motion of an incompressible fluid having constant coefficient of viscosity. So far as we are aware nobody has yet considered the motion of a fluid having varying coefficient of viscosity. In this paper we shall investigate the motion of liquid in which the law of variation in coefficient of viscosity is given by $\mu=\mu_0+\varepsilon_1 x$, for positive values of x. At the origin the value of μ is μ_0 and we shall further assume that μ *is constant and equal to zero for negative values of x.*

In this paper ε_1 has been taken so small that its square and higher powers are neglected and terms of the first order of small quantities only have been retained. Motion of the fluid at a finite distance and at a great distance from the origin has been considered.

The fundamental equations of motion are[1]

$$\rho\,\frac{Du}{Dt} = \rho X + \frac{\partial}{\partial x}p_{xx} + \frac{\partial}{\partial y}\,p_{yx} + \frac{\partial}{\partial z}\,p_{zx}$$

$$\rho\,\frac{Dv}{Dt} = \rho Y + \frac{\partial}{\partial x}\,p_{xy} + \frac{\partial}{\partial y}\,p_{yy} + \frac{\partial}{\partial z}\,p_{zy}$$

$$\rho\,\frac{Dw}{Dt} = \rho Z + \frac{\partial}{\partial x}\,p_{xz} + \frac{\partial}{\partial y}\,p_{yz} + \frac{\partial}{\partial z}\,p_{zz}$$

where X, Y and Z are the extraneous forces and p_{x}, p_{xy}, etc., are stresses[2] given by

$$p_{xx} = -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\,\frac{\partial u}{\partial x}$$

$$p_{yy} = -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\,\frac{\partial v}{\partial y}$$

$$p_{zz} = -p - \tfrac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\,\frac{\partial w}{\partial z}$$

$$p_{yz} = p_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$p_{zx} = p_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$p_{xy} = p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Let $\theta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

Suppose μ be a function of x, y and z so that

$$\frac{\partial}{\partial x}\left(p_{xx}\right) = -\frac{\partial p}{\partial x} - \tfrac{2}{3}\frac{\partial}{\partial x}\left(\mu\,\theta\right) + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial p}{\partial x} - \tfrac{2}{3}\left(\mu\frac{\partial\theta}{\partial x} + \theta\frac{\partial\mu}{\partial x}\right) + 2\left(\mu\frac{\partial^2 u}{\partial x^2} + \frac{\partial\mu}{\partial x}\cdot\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial p}{\partial x} - \tfrac{2}{3}\mu\frac{\partial\theta}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} - \tfrac{2}{3}\theta\frac{\partial\mu}{\partial x} + 2\frac{\partial\mu}{\partial x}\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}\left(p_{yx}\right) = \frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\} = \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial\mu}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial}{\partial z}\left(p_{zx}\right) = \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} = \mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{\partial\mu}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

etc.

Equations of motion then become

$$\rho\frac{Du}{Dt} = \left\{\rho X - \frac{\partial p}{\partial x} + \tfrac{1}{3}\mu\frac{\partial\theta}{\partial x} + \mu\nabla^2 u\right\} + \left[\frac{\partial\mu}{\partial x}\left(2\frac{\partial u}{\partial x} - \tfrac{2}{3}\theta\right) + \frac{\partial\mu}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial\mu}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho\frac{Dv}{Dt} = \left\{\rho Y - \frac{\partial p}{\partial y} + \tfrac{1}{3}\mu\frac{\partial\theta}{\partial y} + \mu\nabla^2 v\right\} + \left[\frac{\partial\mu}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial\mu}{\partial y}\left(2\frac{\partial v}{\partial y} - \tfrac{2}{3}\theta\right) + \frac{\partial\mu}{\partial z}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]$$

$$\rho\frac{Dw}{Dt} = \left\{\rho Z - \frac{\partial p}{\partial z} + \tfrac{1}{3}\mu\frac{\partial\theta}{\partial z} + \mu\nabla^2 w\right\} + \left[\frac{\partial\mu}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{\partial\mu}{\partial y}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \frac{\partial\mu}{\partial z}\left(2\frac{\partial w}{\partial z} - \tfrac{2}{3}\theta\right)\right]$$

If we neglect the inertia terms, the equations of motion of viscous incompressible fluid in the absence of extraneous forces and in the case when μ is simply a function of x, reduce to the forms

$$\frac{\partial p}{\partial x} = \mu \nabla^2 u + \frac{\partial \mu}{\partial x}\left(2\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial p}{\partial y} = \mu \nabla^2 v + \frac{\partial \mu}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial p}{\partial z} = \mu \nabla^2 w + \frac{\partial \mu}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

with $\theta = 0$, *i.e.*, $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

In our case, $\mu = \mu_0 + \varepsilon_1 x$, where μ_0 ~~and~~ ε_1 are constants. Substituting the values of μ and $\dfrac{\partial \mu}{\partial x}$ in the above equations of motion, we get

$$\frac{\partial p}{\partial x} = (\mu_0 + \varepsilon_1 x)\nabla^2 u + \varepsilon_1\left(2\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial p}{\partial y} = (\mu_0 + \varepsilon_1 x)\nabla^2 v + \varepsilon_1\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial p}{\partial z} = (\mu_0 + \varepsilon_1 x)\nabla^2 w + \varepsilon_1\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

with $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

Now we consider two cases when u, v and w are small, the first one being at a finite distance and the other at a great distance from the origin. We shall further assume that ε_1 is small and if we retain terms up to the first order of small quantities, we may neglect $\varepsilon_1\left(2\dfrac{\partial u}{\partial x}\right)$, $\varepsilon_1\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right)$ and $\varepsilon_1\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right)$ from the equations of motion. Then we have to consider whether we can neglect $\varepsilon_1 x \nabla^2 u$, $\varepsilon_1 x \nabla^2 v$ and $\varepsilon_1 x \nabla^2 w$.

When we are taking into account the motion only at a finite distance from the origin, $\varepsilon_1 x \nabla^2 u$, etc. become small quantities of the second order and we may therefore neglect them. The equations of motion then become

$$\frac{\partial p}{\partial x} = \mu_0 \nabla^2 u$$

$$\frac{\partial p}{\partial y} = \mu_0 \nabla^2 v$$

$$\frac{\partial p}{\partial z}=\mu_0\nabla^2 w$$

with

$$\frac{\partial x}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

This case has been solved.[3]

For the motion at a great distance from the origin, μ changes appreciably and we cannot neglect $\varepsilon_1 x\nabla^2 u$, etc., from the equations of motion which become

$$\left.\begin{aligned}\nabla^2 u&=\frac{-1}{\mu_0+\varepsilon_1 x}\frac{\partial p}{\partial x}\\ \nabla^2 v&=\frac{1}{\mu_0+\varepsilon_1 x}\cdot\frac{\partial p}{\partial y}\\ \nabla^2 w&=\frac{1}{\mu_0+\varepsilon_1 x}\frac{\partial p}{\partial z}\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \ldots\ldots \quad (2)$$

Combining (2) with (1), we get

$$\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\frac{1}{\mu_0+\varepsilon_1 x}\nabla^2 p-\frac{\varepsilon_1}{(\mu_0+\varepsilon_1 x)^2}\frac{\partial p}{\partial x}$$

so that the equation for determining 'p' becomes

$$(\mu_0+\varepsilon_1 x)\nabla^2 p-\varepsilon_1\frac{\partial p}{\partial x}=0 \quad \ldots\ldots \quad (3)$$

or

$$\nabla^2 p-\frac{\varepsilon_1}{\mu_0+\varepsilon_1 x}\frac{\partial p}{\partial x}=0.$$

Let $p=RS$, where R is a function of x only and S that of y and z only.

Thus we get

$$\frac{1}{R}\frac{d^2R}{dx^2}-\frac{\varepsilon_1}{\mu_0+\varepsilon_1 z}\cdot\frac{1}{R}\frac{dR}{dx}+\frac{1}{S}\nabla_1{}^2 S=0 \quad \ldots\ldots \quad (4)$$

where

$$\nabla_1{}^2\equiv\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

Equation (4) is clearly satisfied if we write

$$\left.\begin{aligned}&\frac{d^2R}{dx^2}-\frac{\varepsilon_1}{\mu_0+\varepsilon_1 x}\frac{dR}{dx}=KR \quad \ldots\ldots \quad (5)\\ \text{and}\quad &\nabla_1{}^2 S+KS=0 \quad \ldots\ldots \quad (6)\end{aligned}\right\}\text{ where K is an arbitrary constant.}$$

Turning to (5), we get

$$(\mu_0+\varepsilon_1 x)\frac{d^2\mathrm{R}}{dx^2}-\varepsilon_1\frac{d\mathrm{R}}{dx}-\mathrm{K}\left(\mu_0+\varepsilon_1 x\right)\mathrm{R}=0$$

Put $\mu_0+\varepsilon_1 x=\varepsilon_1 x_1$

so that $$\delta x=\delta x_1$$

whereby $$x_1\frac{d^2\mathrm{R}}{dx_1{}^2}-\frac{d\mathrm{R}}{dx_1}-\mathrm{K}x_1\ \mathrm{R}=0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let $\mathrm{R}=\sum_{s=1}^{\infty}\mathrm{A}_s\ x_1{}^{ms}\equiv\mathrm{A}_1\ x_1{}^{m_1}+\mathrm{A}_2x_1{}^{m_2}+\mathrm{A}_3x_1{}^{m_3}+\ .\ .\ .\ +\mathrm{A}_sx_1{}^{ms}+.\ .$

Substituting in (7), we get

$$\Sigma\left\{\mathrm{A}_1m_1\ (m_1-1)\ x_1{}^{m_1-1}-\mathrm{A}_1m_1x_1{}^{m_1-1}\mathrm{A}_1kx_1{}^{m_1+1}\right\}=0$$

or

$$\left.\begin{array}{l} \mathrm{A}_1m_1\ (m_1-2)\ x_1{}^{m_1-1}-\mathrm{A}_1\mathrm{K}x_1{}^{m_1+1} \\ +\ \mathrm{A}_2m_2(m_2-2)x_1{}^{m_2-1}-\mathrm{A}_2\mathrm{K}x_1{}^{m_3+1} \\ +\ \mathrm{A}_3m_3(m_3-2)x_1{}^{m_3-1}-\mathrm{A}_3\mathrm{K}x_1{}^{m_3+1} \\ +\ \quad .\ .\ .\ .\ .\ .\ . \\ +\ \mathrm{A}_{s-1}m_{s-1}(m_{s-1}-2)x_1{}^{ms-1-1}-\mathrm{A}_{s-1}kx_1{}^{ms-1+1} \\ +\ \mathrm{A}_s\ \ m_s\ \ (m_s\ \ -2)x_1{}^{ms-1}-\mathrm{A}_s\ \ kx_1{}^{ms+1} \\ +\ \quad .\ .\ .\ .\ .\ .\ .\ .\ . \end{array}\right\}\equiv 0$$

From this ~~we get~~

$$m_1=2\ or\ zero,\qquad m_1+1=m_2-1\quad\text{or}\quad m_2=m_1+2$$
$$m_3=m_2+2$$
$$=m_1+4$$
$$ms=m_1+2(s-1)$$

and $$\mathrm{A}_{s-1}\ k=\mathrm{A}_s\ m_s\ (ms-2)$$

so that $$\frac{\mathrm{A}_s\ x_1{}^2}{\mathrm{A}_{s-1}}=\frac{\mathrm{K}x_1{}^2}{m_s\ (m_s-2)}=\frac{\mathrm{K}x_1{}^2}{(m_1+2s-2)(m_1+2s-4)}$$

$$\longrightarrow 0\quad\text{as}\quad s\longrightarrow\infty$$

Hence, the series thus found is absolutely and uniformly convergent for all values of x_1.

Since the difference between the two values of m_1 is an integer, *viz.*, 2, the two solutions of R_1 by the method of Frobenius[4] are given by

$$\mathrm{R}_1=\Big[\mathrm{R}\Big]_{m=2}\quad\text{and}\quad\mathrm{R}_2=\left[\frac{\partial\mathrm{R}}{\partial m_1}\right]_{m_1=0}$$

Now $$A_s = \frac{k}{(m_1+2s-2)\ (m_1+2s-4)}\ A_{s-1}, \qquad \text{for } s \geq 2$$

Thus $$A_2 = \frac{k}{(m_1+2)m_1}\ A_1,$$

$$A_3 = \frac{k}{(m_1+4)(m_1+2)}\ A_2 = \frac{k}{(m_1+4)\ (m_1+2)}\ \frac{k}{(m_1+2)m^1}\ A_1,$$

$$A_4 = \frac{k}{(m_1+6)(m_1+4)}\ \frac{k}{(m_1+4)(m_1+2)}\ \frac{k}{(m_1+2)m_1}\ A_1, \quad \text{etc.}$$

$$\therefore \quad R = A_1\ x_1^{m^1}\Big[1+\frac{k}{(m_1+2)\ m_1}\ x_1^2 + \frac{k}{(m_1+4)\ (m_1+2)}\ \frac{k}{(m_1+2)m_1}\ x_1^4 + \frac{k}{(m_1+6)\ (m_1+4)}\ \frac{k}{(m_1+4)\ (m_1+2)}\ \frac{k}{(m_1+2)\ m_1}\ x_1^6 + \ldots\ldots\Big] \quad . \ . \ . \quad (7.1)$$

where A_1 is an arbitrary constant

Since A_1 is an arbitrary constant in order to avoid m_1 in the denominator, we bring in another arbitrary constant B_1 connected by $A_1 = B_1\ m_1$, so that

$$R = B_1\ x_1^{m_1}\Big[m_1 + \frac{k}{m_1+2}\ x_1^2 + \frac{k^2}{(m_1+4)(m_1+2)^2}\ x_1^4 + \frac{k^3}{(m_1+6)\ (m_1+4)^2\ (m_1+2)^2}\ x_1^6 + \ldots\Big] \quad . \ . \ . \quad (7.2)$$

$$\frac{\partial R}{\partial_{m_1}} = R \log x_1 + B_1 x_1^{m_1}\Big[1-\frac{kx_1^2}{(m_1+2)^2} - \frac{k^2 x_1^4}{(m_1+4)(m_1+2^2)}\left(\frac{1}{m_1+4}+\frac{2}{m_1+2}\right) - \frac{k^3 x_1^6}{(m_1+6)\ (m_1+4)^2\ (m_1+2)^2}\left(\frac{1}{m_1+6}+\frac{2}{m_1+4}+\frac{2}{m_1+2}\right) - \ldots\ldots\Big] \quad . \ . \ . \quad (7.3)$$

Thus

$$R_1 = B_1\ x_1^2\Big[\ 2+\frac{k}{4}\cdot\ x_1^2 + \frac{k^2}{6.4^2}x_1^4 + \frac{k^3}{8.6^2 4^2}\ x_1^6 + . \quad .\Big] \quad . \ . \quad (7.4)$$

$$R_2 = B_1 \log x_1\Big[\frac{k}{2}\ x_1^2 + \frac{k^2}{4.2^2}\ x_1^4 + \frac{k^3}{6.4^2 2^2}\ x_1^6 + \ . \ .\Big] + B_1\Big[\ 1-\frac{k}{2^2}\ x_1^2 - \frac{k^2 x_1^4}{4.2^2}\left(\tfrac{1}{4}+\tfrac{2}{2}\right) - \frac{k^3 x_1^6}{6.4^2.2^2}\left(\tfrac{1}{6}+\tfrac{2}{4}+\tfrac{2}{2}\right) - \ldots\Big] \quad . \quad (7.5)$$

Then we consider

$$(\nabla_1^2 + k)\ S = 0 \quad . \ . \ . \ . \ . \ . \ . \quad (6)$$

Now k may be negative or positive.

Case (i). $\qquad k = \beta_1^2$

Let $y = \omega \cos \phi$ and $z = \omega \sin \phi$.

The equation (6) becomes

$$\frac{\partial^2 S}{\partial \omega^2} + \frac{1}{\omega}\frac{\partial S}{\partial \omega} + \frac{1}{\omega^2}\frac{\partial^2 S}{\partial \omega^2} + \beta_1{}^2 S = 0 \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

If S to be finite for $\omega = 0$, the solution of (6.1)[5] is given by

$$S = B^1{}_n\, J_n\left(\beta_1\, \omega\right) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} n\,\phi$$

where n may have any of the values 0, 1, 2, 3, ... and $B^1{}_n$ is an arbitrary constant and J_n is Bessel's function of the n^{th} order.

Case (ii). $k = -\beta_2{}^2$

Then equation (6) reduces to

$$\frac{\partial^2 S}{\partial \omega^2} + \frac{1}{\omega}\frac{\partial S}{\partial \omega} + \frac{1}{\omega^2}\frac{\partial^2 S}{\partial \phi^2} + \beta_2{}^2 S = 0 \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

If S is to be finite for $\omega = 0$, the solution of (6.2)[6] is given by

$$S = B^{11}{}_m\, I_m\,(\beta_2\, \omega) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\phi$$

where m may have any of the values 0, 1, 2, 3, ... and $B^{11}{}_m$ is an arbitrary constant and where

$$I_m\,(z) = \frac{z^m}{2^m m!}\left\{1 + \frac{z^2}{2\,(2m+2)} + \frac{z^4}{2.4\,(2m+2)\,(2m+4)} + \cdot \quad \cdot \quad \cdot\right\}$$

If $k = 0$, we get

$$\frac{d^2 R_0}{dx^2} - \frac{\varepsilon_1}{\mu_0 + \varepsilon_1 x}\;\frac{dR_0}{\partial x} = 0 \;\&\; \nabla_1{}^2 S_0 = 0$$

or $\log \dfrac{dR_0}{\partial x} = \log\left\{A_0\,(\mu_0 + \varepsilon_1 x)\right\}$, where A_0 is an arbitrary constant

or $$\frac{dR_0}{dx} = A_0\,(\mu_0 + \varepsilon_1 x)$$

or $R_0 = \dfrac{A_0}{2\varepsilon_1}(\mu_0 + \varepsilon_1 x)^2 + C = A\,(\mu_0 + \varepsilon_1 x)^2 + C$, where $\dfrac{A_0}{2\varepsilon_1} = A$ and C is another arbitrary constant.

If we take c to be zero, $R = A\,(\mu_0 + \varepsilon_1 x)^2$.

Now turning to our original equation and substituting $p = RS$ and $\mu_0 + \varepsilon_1 x = \varepsilon_1\, x_1$, we get

$$\nabla^2 u = \frac{S}{\varepsilon_1 x_1}\;\frac{dR}{dx_1}$$

$$\nabla^2 v = \frac{R}{\varepsilon_1 x_1}\;\frac{\partial S}{\partial y}.$$

$$\nabla^2 w = \frac{R}{\varepsilon_1 x_1} \frac{\partial S}{\partial z}$$

with $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, where $x_1 \frac{d^2R}{dx_1^2} - \frac{dR}{dx_1} - kx_1 R = 0$ and $(\nabla_1{}^2 + k) S = 0$

A particular solution is obtained if we write

$$\left.\begin{aligned} u &= kS \int V\, dx_1 \\ v &= V \frac{\partial S}{\partial y} \\ w &= V \frac{\partial S}{\partial z}. \end{aligned}\right\} \qquad (8)$$

where V is a function of x_1 given by $\frac{d^2V}{dx_1{}^2} - kV = \frac{R}{\varepsilon_1 x_1}$ . . . . . . (9)

for $$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = k\, S\, V + V\nabla_1{}^2S = V\, (\nabla_1{}^2 + k)\, S = 0.$$

and $$\nabla^2 v = V \frac{\partial}{\partial y} \nabla_1{}^2 S + \frac{\partial S}{\partial y} \frac{d^2V}{dx_1{}^2}$$

$$= -k\, V \frac{\partial S}{\partial y} + \frac{dS}{\partial y} \frac{d^2V}{dx_1{}^2} = \frac{\partial S}{\partial y}\left(\frac{d^2V}{dx_1{}^2} - kV\right) = \frac{R}{\varepsilon_1 x_1} \frac{\partial S}{\partial y}, \text{ by (9)}$$

Similarly, $\nabla^2 w = \frac{R}{\varepsilon_1\, x_1} \frac{\partial S}{\partial z}$

$$\nabla^2 u = k\int V\, dx_1, \nabla_1{}^2 S + k\, S \frac{dv}{dx_1} = kS\left(\frac{dV}{dx_1} - k\int V\, dx_1\right)$$

so that $k\left(\frac{dV}{dx_1} - k\int V\, dx_1\right)$ must be equal to $\frac{1}{\varepsilon_1\, x_1} \frac{dR}{dx_1}$ in order that our solution may be valid.

Or, $$\frac{d^2V}{dx_1 2} - kV = \frac{1}{\varepsilon_1 k}\left(\frac{1}{x_1} \frac{d^2R}{dx_1{}^2} - \frac{1}{x_1{}^2} \frac{dR}{dx_1}\right) = \frac{kx_1 R}{k\varepsilon_1 x_1{}^2} = \frac{R}{\varepsilon_1 x_1},$$ which holds.

Hence $k\left(\frac{dV}{dx_1} - k\int V dx_1\right)$ may differ from $\frac{1}{\varepsilon_1 x_1} \frac{dR}{dx_1}$ by an arbitrary constant. Choosing this arbitrary constant to be zero, we shall get the required result.

When k is zero, we have

$$\nabla^2 u = 2A\varepsilon_1\, S_0$$

$$\nabla^2 v = A\varepsilon_1 x_1 \frac{\partial S_0}{\partial y}$$

$$\nabla^2 w = A\varepsilon_1 x_1 \frac{\partial S_0}{\partial z}.$$

with $$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{where } \nabla_1{}^2 S_0 = 0$$

Here a particular solution is given by

$$\left.\begin{aligned} u &= f(y_1 z) \\ v &= \frac{x_1^3}{6}\cdot\frac{\partial S_1}{\partial y} \\ w &= \frac{x_1^3}{6}\cdot\frac{\partial S_1}{\partial z} \end{aligned}\right\} \text{Such that } \nabla_1^2 f(y_1 z) = 2S_1 \text{ where } S_1 = A\varepsilon_1 S_0.$$

Now we consider the values of u_1, v_1 and w_1 when u_1, v_1 and w_1 are given by

$$\nabla^2 u_1 = 0,\ \nabla^2 v_1 = 0, \text{ and } \nabla^2 w_1 = 0$$

with
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

These give[7] u_1, v_1 and w_1 to be

$$u_1 = \sum\left(\frac{\partial\phi_n}{\partial x} + z\frac{\partial\chi_n}{\partial y} - y\frac{\partial\chi_n}{\partial z}\right)$$

$$v_1 = \sum\left(\frac{\partial\phi_n}{\partial y} + x\frac{\partial\chi_n}{\partial z} - z\frac{\partial\chi_n}{\partial x}\right)$$

$$w_1 = \sum\left(\frac{\partial\phi_n}{\partial z} + y\frac{\partial\chi_n}{\partial x} - x\frac{\partial\chi_n}{\partial y}\right)$$

where the harmonics ϕ_n and χ_n are arbitrary.

$\therefore$ A complete solution of (1) & (2) is given by

$$\left.\begin{aligned} u &= ks\int V dx + \Sigma\left(\frac{\partial\phi_n}{\partial x} + z\frac{\partial\chi_n}{\partial y} - y\frac{\partial\chi_n}{\partial z}\right) \\ v &= V\frac{\partial s}{\partial y} + \Sigma\left(\frac{\partial\phi_n}{\partial y} + x\frac{\partial\chi_n}{\partial z} - z\frac{\partial\chi_n}{\partial x}\right) \\ w &= V\frac{\partial s}{\partial z} + \Sigma\left(\frac{\partial\phi_n}{\partial z} + y\frac{\partial\chi_n}{\partial x} - x\frac{\partial\chi_n}{\partial y}\right) \end{aligned}\right\} \quad . \ . \quad (10)$$

where S and V are given by (6) and (9)

Also, if we denote by ξ, η and ζ, the components of vorticity, we find

$$\left.\begin{aligned} \xi &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \Sigma(n+1).\ \frac{\partial\chi_n}{\partial x} \\ \eta &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial s}{\partial z}\left(k\int v dx - \frac{dV}{dx}\right) + \Sigma(n+1)\frac{\partial\chi_n}{\partial y}. \\ \zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial S}{\partial y}\left(\frac{dv}{dx} - k\int V dx\right) + \Sigma(n+1)\frac{\partial\chi_n}{\partial z} \end{aligned}\right\} \quad . \ . \quad (11)$$

By (10)

$$xu + yv + zw = kSx\int V dx + V\left(y\frac{\partial S}{\partial y} + z\frac{\partial S}{\partial n}\right) + \Sigma n\phi_n \quad . \ . \ . \ . \quad (12$$

The components of stress across the surface of a sphere of radius r are given by[8]

$$p_{rx} = \frac{x}{r}\ p_{xx} + \frac{y}{r}\ p_{xy} + \frac{z}{r}\ p_{xz} \qquad \ldots \ldots \ldots \quad (13)$$

Substituting the values of p_{xx}, p_{xy}, p_{xz}, ... we get

$$\left.\begin{aligned} rp_{rx} &= -\,xp+\mu\left(r\,\frac{\partial}{\partial r} - 1 \right)u+\mu\,\frac{\partial}{\partial x}\,(xu+yv+zw) \\ rp_{ry} &= -\,yp+\mu\left(r\,\frac{\partial}{\partial r} - 1 \right)v+\mu\,\frac{\partial}{\partial y}\,(xu+yv+zw) \\ rp_{rz} &= -\,zp+\mu\left(r\,\frac{\partial}{\partial r} - 1 \right)w+\mu\frac{\partial}{\partial z}\,(xu+yv+zw) \end{aligned}\right\} \quad \ldots \quad (14)$$

Replacing the values of p, u, v, w, μ and $xu+yv+zw$, we shall get the required expressions for the stresses.

Now we have to find the value of V from (9) which reads

$$\frac{d^2\mathrm{V}}{dx_1^{\,2}} - \mathrm{KV} = \frac{\mathrm{R}}{\varepsilon_1 x_1}$$

The complementary function of (9) is given by

$$\mathrm{V} = \mathrm{C}_1\, e^{\sqrt{k}\,x_1} + \mathrm{C}_2\, e^{-\sqrt{k}\,x_1}$$

The particular integral is given by

$$\begin{aligned} \mathrm{V} &= \frac{1}{\mathrm{D}^2-k}\cdot\frac{\mathrm{R}}{\varepsilon_1 x_1}. \\ &= \frac{1}{2\sqrt{k}\epsilon_1}\left(\frac{1}{\mathrm{D}-\sqrt{k}} - \frac{1}{\mathrm{D}+\sqrt{k}}\right)\mathrm{R}. \\ &= \frac{1}{2\sqrt{k}\,\varepsilon_1}\left\{ e^{\sqrt{k}x_1}\int e^{-\sqrt{k}x_1}\,\mathrm{R}\ dx_1 - e^{-\sqrt{k}x_1}\int e^{\sqrt{k}x_1}\,\mathrm{R}\,dx_1 \right\} \end{aligned}$$

This expression shows that we have to evaluate integrals of the forms

$$\int x^a\, e^{ax}\,dx \text{ if } k \text{ is positive or } \int x^a \left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} bx\,dx \text{ if } k \text{ is negative}$$

and $$\int x^a\ e^{az}\log x\,dx \quad\text{or}\quad \int x^a \log x. \ \Big\}\ bx\,dx$$

But all the above forms are integrable. Thus the solution of V is obtained.

My best thanks are due to Prof. A. C. Banerji, under whom I have worked out this problem.

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MAXIMUM USABLE FREQUENCIES FOR SKY-WAVE TRANSMISSION

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Communicated by Dr. D. S. Kothari

(Received on July 5, 1941)

Introduction

The propagation of radio waves depends upon a number of factors chief among which are the frequency used, the angle of radiation of the transmitting aerial, the time of the day or night, the distance to be covered, the latitude of the place. and the prevailing sun-spot activity.

The graphs of maximum usable frequencies for transmission of radio waves have been published in the Report of the Committee on Radio Wave Propagation,[1] but these are drawn on the basis of measurements of the ionospheric heights and critical frequencies for normal incidence and refer to a particular latitude (Washington D. C. 39° 48′ N.). Methods of obtaining maximum usable frequencies taking into account the earth's magnetic field and the curvature of the ionosphere have been presented by N. Smith,[2] and the graphs of maximum usable frequencies have been re-drawn by Gilliland, Kirby, Smith and Reymer,[3] for Washington taking these factors into considerations. Appleton and Beynon[1] have calculated the multiplying factors for various distances required to obtain the maximum usable frequencies from data of critical frequencies at normal incidence taking into consideration the curvature of the earth, and assuming a parabolic distribution of the ionization density in the ionosphere, near the point of maximum density, as shown in Fig. 3.

Various attempts have been made to co-relate sunspot numbers with maximum usable frequencies to find out some law governing their relationships and important among these are those given by Durkee,[5] Kosikov[6] and Gromov and Young[7] and Hulburt.

In this paper the graphs of maximum usable frequencies for Washington have been re-drawn in the light of Appleton and Beynon's corrections figures 1, 2, and from these graphs by the application of the Chapman's theory the values of maximum usable frequencies for other latitudes (in this instance Delhi) have been deduced. Further the effect of sunspot numbers on the graphs has been studied and new graphs prepared up to the next sunspot minimum, *i.e.*, 1944. An attempt has also been

made to show how far these graphs agree with observed facts and thus a method is suggested for predicting the maximum usable frequencies.

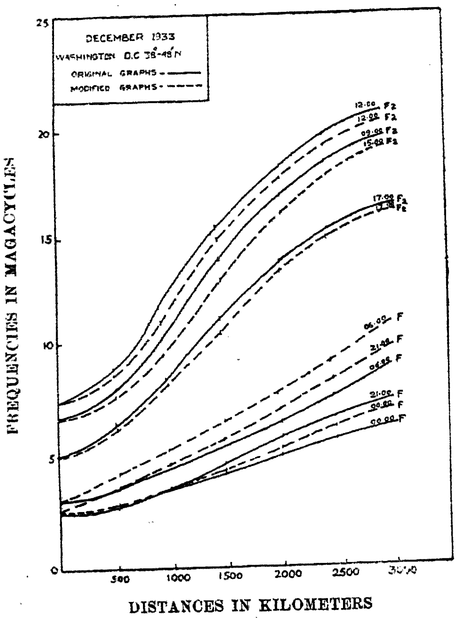


Fig. 1. Graphs showing the original and modified values of maximum usable frequencies for various distances of transmission. The numbers on the graphs refer to Washington local time and F or F_2 indicates the layer from which reflection takes place at that time.

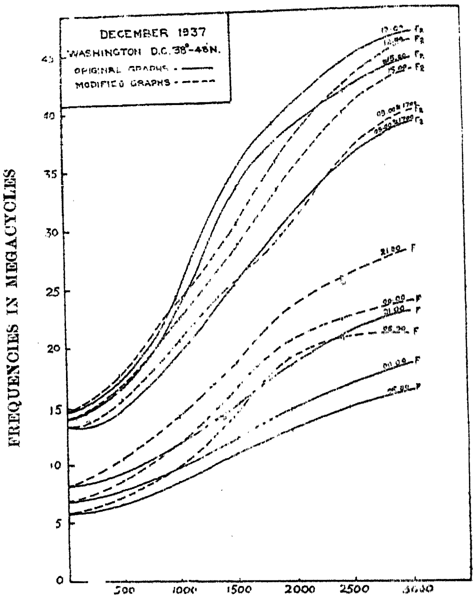


Fig. 2. Graphs showing the original and modified values of the maximum usable frequencies for various distances of transmission. The numbers on the graphs refer to Washington local time and F or F_2 indicates the layer from which reflection takes place at that time.

For accurate determination of maximum usable frequencies the following important factors must be taken into consideration :—

(1) The variation of the virtual height of the ionosphere near critical frequency.

(2) Earth's Magnetic Field.

(3) Curvature and thickness of the ionosphere.

(4) Latitude of the place.

(5) The angle of radiation or departure from the aerial.

(6) Sunspot activity.

(1) This difficulty for the case of flat ionosphere has been solved by N. Smith.[8]

(2) The effect of earth's magnetic field is to split the wave into two polarised components known as ordinary and extra-ordinary rays. The maximum usable frequencies for the extra-ordinary ray is always

greater than for the ordinary ray and varies from place to place depending upon earth's magnetic field. Except for short distances, the difference is not great and decreases with increasing frequency and distance. For Delhi this difference has been calculated by the author to be ·65 Mc/S. For practical communication problems it is sufficiently accurate to make the calculations for the ordinary ray.

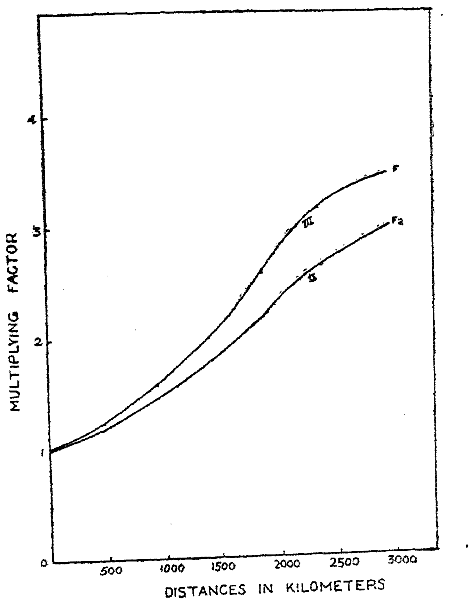


Fig. 3. Graphs showing the values of multiplying factor for various distances of transmission for F and F_2 layer when the parabolic nature and thickness of the ionosphone and curvature of the earth are taken into consideration.

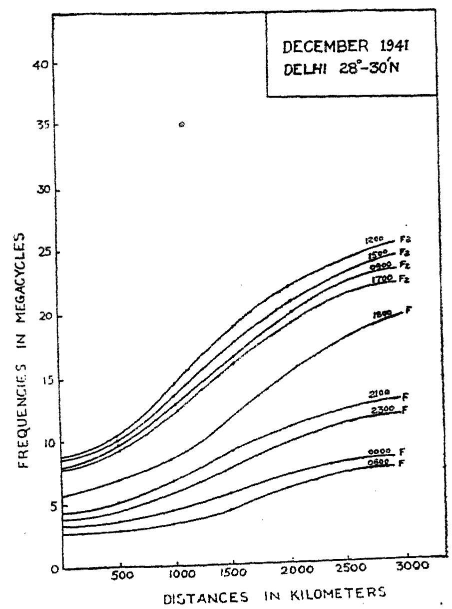


Fig. 4. Graphs showing the calculated values of maximum. usable frequencies for various distances of transmission. The numbers on the graphs refer to I. S. T. and F or F_2 indicates the layer from which reflection takes place at that time.

(3) The usual graphs are drawn on the assumption that the ionosphere is flat. Appleton and Beyon[4] have calculated multiplying factors which enable the graphs to be modified to take into consideration the thickness and parabolic distribution of electronic density in the ionosphere at least near the region of maximum ionization density which is the case as shown by Appleton.[9] On the basis of these data the graphs for Washington have been re-drawn for two different cases, *viz.*, when the flatness of ionosphere and curvature of the earth are taken into consideration and when thickness and parabolic nature of the ionosphere are also taken into consideration. It is found that the

values of multiplying factors are greater for both the layers, *i.e*, F and F_2 when the ionosphere is taken to be flat than when it is taken to be 'parabolic'. The values of maximum usable frequencies for various distances are obtained by multiplying the values of critical frequencies for normal incidence with the corresponding values of the multiplying factors.

Effect of the Latitude

(4) The maximum usable frequency varies in accordance with the variation of ionization with latitude which agrees fairly well with the predictions of Chapman's[10] theory, ordinarily the ionization decreases as we proceed away from the equator towards higher latitude. Appleton[11] and Naismith have found on comparison that the daily maximum ionization of the E layer at Washington (Lat. 38° 48′ N.) is greater than that at London (Lat. 51° 30′ N.) under normal conditions. Observations at Huancayo[12] show that average F_2 ionization is much greater at Huancayo (Lat. 12° 17′ S.) than at Washington.

The variation of ionization with height and zenith distance of the sun (which depends upon the season, latitude and time at the place in consideration) has been dealt by E. Chapman[10] on the assumption that the density of the atmosphere varies exponentially with height and ionization is caused by absorption of mono-chromatic light. He further assumes that the ionized products re-combine with one another and are assumed not to diffuse away. Taking into account the variation of the rate of ionization due to the rotation of the earth and assuming a constant re-combination coefficient, the distribution of density of ionized products is determined. The rate of production of ions I per c. c. by mono-chromatic light at a height h when incident at an angle x is given by

$$I = I_0 \, e^{1 - z - e^{-z \operatorname{Sec} x}} \qquad (1)$$

where z reduced height

$$= \frac{h - h_0}{H}$$

$I_0 = I$ when $x = 0$

h_0 = the height of the level above the equator at noon when $x = 0$

H = Scale height $= \frac{KT}{mg}$

T = temperature of atmosphere
K = Boltzman's constant
m = mass of a molecule of air
g = acceleration due to gravity

In wireless experiments and calculations we are not concerned with the rate of ionization so much as with N. the ionic density.

Hence, $\frac{dN}{dt}=I-\alpha N^2$ . . . . (2)

where α is the re-combination coefficient t is time in seconds.

Hence I (which is supposed to be zero during night) is, during the day, a function of the height h (or z), of the time of the day (t or ϕ), of the co-latitude θ, of the season (through the sun's declination δ). $\propto$ may also vary but has been taken to be constant. From equation (2) methods of obtaining numerical values of N/No have been developed by Chapman[10] and Millington.[13]

Suppose N_A is the noon maximum ionization at place A

" N_B " " " " at place B

θ_A is the co-latitude of place A

θ_B is the co-latitude of place B

δ_A is the sun's declination at place A

δ_B is the sun's declination at place B

Then, from Chapman's heory,

$$\frac{N_A}{N_B}=\sqrt{\frac{\operatorname{Sin}(\theta_A+\delta_A)}{\operatorname{Sin}(\theta_B+\delta_B)}} \quad \ldots \ldots \quad (3)$$

Also $N \propto (\cos x)^{\frac{1}{2}}$ where x is sun's zenith distance.

and $\therefore f \propto (\cos x)^{\frac{1}{4}}$ where f, is Ho critical frequency.

Suppose f_A is the critical frequency at place A

" f_B " " at place B

$$\therefore \frac{f_A}{f_B}=\left\{\frac{\cos x_A}{\cos x_B}\right\}^{\frac{1}{4}} \quad \ldots \ldots \quad (4)$$

On the equinoxes and at noon

$$x=\frac{\pi}{2}-(\theta+\delta)$$

$$\therefore \frac{f_A}{f_B}=\left\{\frac{\sin(\theta_A+\delta_A)}{\sin(\theta_B+\delta_B)}\right\}^{\frac{1}{4}} \quad \ldots \ldots \quad (5)$$

As pointed before, Chapman's simple theory does not take the following important factors into consideration :—

(*a*) The absorption of several spectral bands of ultraviolet light having different absorption co-efficients.

(*b*) Changes in atmospheric temperature, with height and time.
(*c*) Phenomena of diffusion and electron attachment.
(*d*) Variation of the re-combination co-efficient α
(*e*) The selective absorption of the ultra-violet light by the different layers of the atmosphere.

Hence the variation of ionic density with height and time derived from equation (2) holds good only for the case of F and F_1 layers as verified experimentally by Appleton and Naismith.[11] Also equation (5) is true only for the equinoxes and at the noon.

As the region F_2 behaves abnormally, many theories have been put forward to explain this anomaly. That given by Appleton[14] seems to be more accurate. He holds the view that this apparent abnormality is due to the fact that the molecular temperature of the outermost layers of the atmosphere is greatly increased in summer causing a lowering of ion production on the one hand and volume expansion on the other and the maximum ionization density is thus effectively reduced. In support of this, Appleton and Naismith[15] have found experimentally on the time of early morning of minimum ionization that the minimum occurs near about ground sunrise in summer but about an hour earlier in winter. On the basis of this theory separate curves are drawn to show the variation of N with height for F_2 layer in accordance with the following equation :—

$$\frac{dN}{dt}=I-\frac{N}{T}\frac{dT}{dt}-\alpha N^2 \quad \ldots \quad (6)$$

In equation (6), the first term expresses the rate of ion-production by solar radiation, and the second term expresses the rate of dilution or concentration of electrons, ions and molecules resulting from thermal expansion or contraction due to solar heating.

Equation (5) can be modified to be applicable to all latitudes, seasons of the year and times of the day. The sun's zenith distance depends on the season, and on the latitude and time of the day at the point considered. Time 't' in seconds reckoned from noon when expressed in angular measure ϕ becomes as below :—

$$t=\frac{86400\phi}{2\pi}=1{\cdot}37\times 10^4\phi \quad \ldots \quad (7)$$

Since the meridian through the sun corresponds to local noon ($\phi=0$) sun's zenith distance x is given by

$$\cos x=\sin\delta\cos\theta+\cos\delta\sin\theta\cos\phi \quad \ldots \quad (8)$$

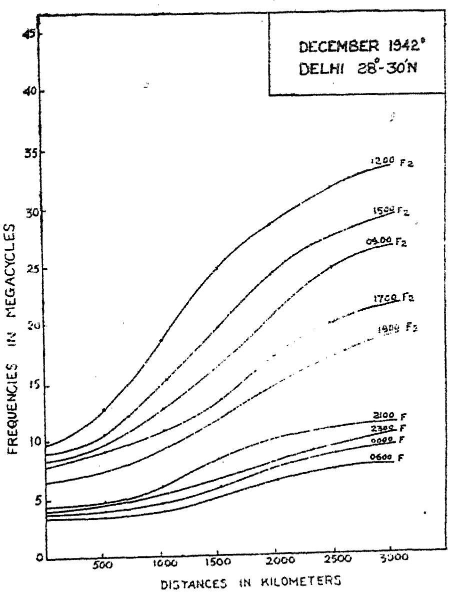
Hence equation (5) for F_2 layer becomes as given below :—

$$\frac{f_A}{f_B}=\left\{\frac{\sin\delta_B\cos\theta_B+\cos\delta_B\sin\theta_B\cos\phi_B}{\sin\delta_A\cos\theta_A+\cos\delta_A\sin\theta_A\cos\phi_A}\right\}^{\frac{1}{4}} \quad \ldots \quad (9)$$

where the symbols have their previous meanings and x has been given the most general value for all the timings and latitudes.

Curves drawn from the equation (6) give the variation of the rate of ion-production levels of N max with height for different zenith distances of the sun and from these the ratio of maximum ionic density for two places for the same season and time can be found out. Thus the ratio of maximum usual frequencies for the two places can be determined. As it is more convenient to express the ratio of the maximum usable frequencies in terms of declination δ, equation (9), which gives nearly the same values is more useful and convenient for the purposes of calculations.

(5) The effect of angle of radiation has been studied in greater detail and has formed the subject of a separate paper by the author.



F. 5. Graphs showing the calculated values of maximum usable frequencies for various distances of transmission. The numbers on the graphs refer to I. S. T. and F or F_2 indicates the layer from which reflection takes place at that time.

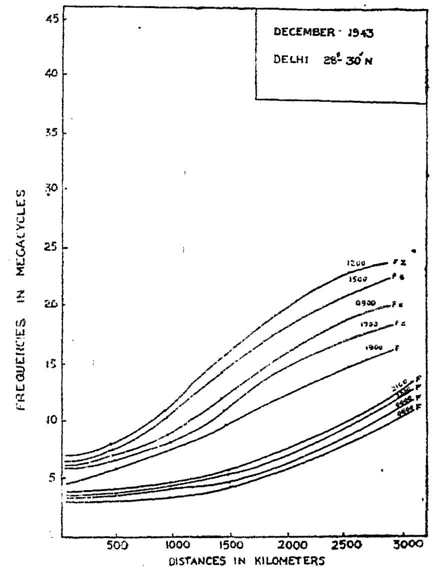


Fig. 6. Graphs showing the calculated values of maximum usable frequencies for various distances of transmission The numbers on the graphs refer to I. S. T and F or F_2 indicates the layer from which reflection takes place at that time.

(6) *Effect of the sunspot activity.*—The average equivalent electron density of the ionosphere shows a variation corresponding to 11·11 year cycle of the sunspot activity. Till recently no attempts were made to correlate the critical frequencies with sunspot numbers though it was indicated by various workers [16, 17, 18, 19, 20] that critical frequencies for F, F_1, F_2 layers increased with increasing sunspot

activity from 1928 the year of last sunspot maximum activity to 1933 the year of last sunspot minimum activity.

Young and Hulburt[7] have correlated the sunspot numbers with the critical frequencies from 1923 to 1936. The formula (though arbitrary) given by them is

$$f=a(s+b)^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Where f is the critical frequency and S is the the sunspot number and a and b are constants. They found that $a=7{\cdot}8$ and $b=12$ were the values that agreed with the experimental observations. But the evidence derived from critical frequency measurements of region F_2 at vertical incidence cannot be regarded as reliable until 1933 when the distinction between the ordinary and extra-ordinary ray was realized for the first time. The monthly critical frequencies from the month of December 1937 to December 1940 for Washington as given in P.I.R.E. journals and corresponding monthly sunspot numbers from Wolf and Wolfer's monthly

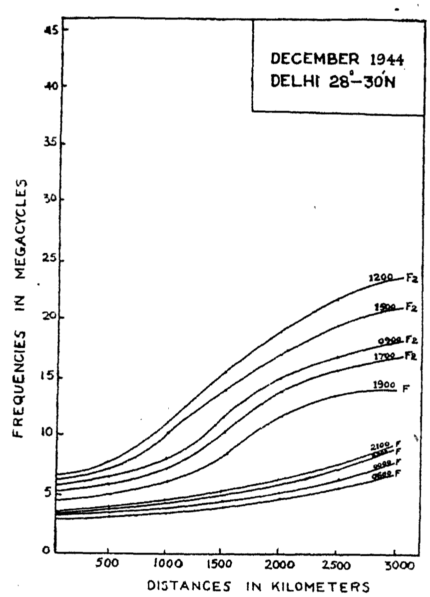


Fig. 7. Graphs showing the calculated values of maximum usable frequencies for various distances of transmission. The number on the graphs refer to I. S. T. and F or F_2 indicates the layer from which reflection takes place at that time.

reviews have been taken by the author and they are found to obey the following law very approximately :—

$$f\alpha(s \pm b)^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where the symbols have the meanings as in equation (10).

It is found that "b" is not constant but varies from 10·4 to 16·5 throughout the complete cycle and is positive in ascending and negative in the ascending and descending portions of the cycle.

The values of maximum usable frequencies for Delhi have been calculated from 1941 to 1944 for the months of December only taking into consideration the effect of the sunspot activity and latitude as shown graphically in Figures 4, 5, 6 and 7.

CONCLUSION

From the graphs for December 1941, it is seen that the maximum usable frequencies for $0090F_2$, $1200F_2$, 2100F, for a skip distance of 500 K.M. are 9.85 mgs., 10.8 mgs. and 5.4 mgs. respectively.

These values compare favourably well with the actual frequencies used at Delhi when it is considered that the optimum frequencies used are always less by 15% than the maximum usable frequency obtained theoretically in order to compensate the variation in the ionospheric conditions from time to time, because in determining the utility of a given radio frequency for a given time and path a complete specification includes data on the wave absorption as well.

By actual pulse measurements at various times from November 1 to February 28, 1940-41, the Research Department of All-India Radio has predicted the frequencies for the winter (1941-42) which agrees well with the values given in the graphs for December 1941. But here for the purposes of comparison values of frequencies which are actually in use have been taken, as the predicted values have undergone slight modifications.

Transmissions	Call signs of transmitters	Skip-distances	Frequencies actually in use	Frequencies as calculated
Transmission I ...	Vu D_2	Zero K.M.	7.3 Mgs.	7.8 Mgs.
(08.00 to 10.00)... ...	Vu D_3	1000 K.M. at 08.00	11.83 Mgs	12.5 Mgs.
	Vu D_4	500K.M at 08.00	9.59 Mgs.	9.85 Mgs.
Transmission II ...	Vu D_2	Zero K. M.	7.3 Mgs.	8.4 Mgs.
(12.00 to 14.00) ...	Vu D_3	1000 K.M. at 12.00	15.29 Mgs.	17.20 Mgs,
	Vu D_4	500 K.M. at 12.00	9.59 Mgs.	10.8 Mgs.
Transmission III ...	Vu D_2	Zero K.M.	4.96 Mgs.	6.0 Mgs.
Part I (16.30 to 19.00) ...	Vu D_3	1600 K.M. at 19.00	11.83 Mgs.	11.5 Mgs.
	Vu D_4	1000 K.M. at 19.00	9.59 Mgs.	10.2 Mgs.
Transmission III ...	Vu D_2	Zero K.M.	3.5 Mgs.	4.0 Mgs.
(19.00 to 23.00) ...	Vu D_3	1500 K.M. at 23.00	6.13 Mgs.	6.0 Mgs.
	Vu D_4	2000 K.M. at 23.00	9.59 Mgs.	8.1 Mgs.

In those cases where it does not agree well, it does not show that the values are not correct. As it is not convenient to change the frequencies from hour to hour some compromise in the transmission frequencies becomes necessary. Taking these difficulties in view it has been suggested by the author in a separate paper to affect a change in the maximum usable frequencies as well as the angles of radiation so that changes in the ionosphere for a longer period can be compensated.

Thus the values of maximum usable frequencies for any place for single hop transmission can be predicted with sufficient degree of accuracy. For distances involving more than a single hop transmission, knowledge of either the down-coming angle at the receiving end or the angle of radiation at the transmitting end is essential. Having determined the order of the hop, the maximum usable frequency for that distance can easily be calculated but in no case the maximum usable frequency wiil be more than that for the single hop as long there is no change in the angle of radiation or down-coming angle. As for example the maximum usable frequency for a distance of 500 K.M. by a single hop transmission will be the same as for 1000 K.M. by double hop or 1500 K.M. by third hop.

ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. Wali Mohammad under whose guidance this paper was prepared and who took the pains of correcting and re-correcting it. The author also wishes to thank the Chief Engineer, All-India Radio, who has very kindly given the permission to make use of the table of forecast of frequencies for winter 1941-1942.

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CHEMICAL EXAMINATION OF THE SEEDS OF *NIGELLA SATIVA*, LINN. (*MAGREL*) PART I. FATTY OIL

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SUMMARY

The fatty oil from the seeds of *Nigella sativa* Linn. (Magrel) has been examined and found to contain the glycerides of oleic, linolic, myristic, palmitic and stearic acids, the percentage of which are given below:—

Oleic	44·45
Linolic	35·99
Myristic	0·26
Palmitic	6·31
Stearic	2·45
Unsaponifiable (sterol)	0·03

It has been shown that the oil examined by Crossley and Le Sueur (*loc. cit.*) was not a volatile oil but a mixture of fatty and volatile oils (Table 1). Further the diolefinic acid present in the oil of *Nigella sativa* is linolic acid in our case, whereas it is telfairic acid in the oil examined by Bures and Mladkova (*loc. cit.*).

Further work on the examination of the other constituents of the seeds of *Nigella sativa* is in progress.

Nigella sativa, commonly known as small fennel in English, Krishnajira in Sanskrit and Kalajira, Kalaunji or Magrel in Hindustani, is an annual herb belonging to the natural order Ranunculaceæ. According to G. Watt,[1] it is cultivated in many parts of India for its seeds. I. H. Burkle,[2] however, says, "It is rare to find it away from the extreme North-Western parts of this country. Nor does the local production supply even the Punjab market for there is an import of the seeds into India from Afghanistan, beyond which is the home of its cultivation.*" The seeds are small, triangular in shape, resemble gunpowder in appearance and possess a sharp bitter taste.

A proximate analysis of the seeds was made by Greenish[3] who found that they contain fixed oil (37·1%), volatile oil (1·64%), albumen and sugars. A preliminary

* Dr. N. L. Bor of Dehradun (Private communication) writes as follows:—

Nigella sativa is occasionally cultivated in the plains of India and is often found as an escape. It is not known whether the local production is sufficient for the needs of the country as it is extensively used as a condiment as well as medicinally.

examination of the oil from the seeds has been done by Bures and Mladkova[4] whose results are recorded in Table I. These authors[5] later examined another sample of the oil and reported the presence of myristic and telfairic acids. Their results are given in Table I. Kh. Dorn and R. M. Erastova[6] found that it is a good film-forming oil when applied as lacquer after polymerisation. Max Roberg[7] found that *Nigella sativa* seeds contain a saponine and later M. Roberg and E. Marchal[8] showed that the different lots of seeds showed a variable saponine content.

G. Watt[9] reported the presence of two kinds of oil in the seeds—one dark-coloured fragrant and volatile oil and the other clear, nearly colourless and about the consistency of castor oil. Crossley and Le Sueur[10] examined a sample of the oil from the seeds of *Nigella sativa* which they said was dark reddish brown in colour, semi-drying in nature and had a marked odour of eucalyptus oil. They claimed their sample to be a volatile oil. Their results are recorded in Table I. Brodie[11] says that the oil examined by Crossley and Le Sueur must have been either a fatty oil or a mixture of fatty and volatile oils and not purely a volatile oil and hence the problem needs further elucidation.

In view of the great medicinal value of the seeds as reported by Dey,[12] Dymock,[13] and Kirtikar and Basu[14] and the discrepancy noted by Brodie (*loc. cit.*) regarding the nature of the oil, it was thought desirable to make a thorough chemical examination of the oil from the seeds, and in the present communication we have examined the fatty oil from the seeds of Indian origin, a detailed chemical examination of which has not been done so far.

The physical and chemical constants of the oil extracted with benzene from the seeds of *Nigella sativa* as found by us are given in Table I. The oil obtained on extraction with benzene has been steam distilled to free it from any volatile oil and the physical and chemical constants of the pure fatty oil thus obtained are also recorded in Table I. The fatty oil so obtained was clearer than the original oil and had a dark reddish brown colour. It was, however, not clear and nearly colourless as stated by Watt (*loc. cit.*). The volatile oil was present only to the extent of 0·4 per cent. in the oil obtained by extraction with benzene. It was dark reddish in colour and had a very sharp odour. A comparison of the physical and chemical constants of the oil before and after steam distillation (Table I) shows that the differences are slight. This is due to the fact that the amount of volatile oil is too small. It will also be seen that the data of Crossley and Le Sueur (*loc. cit.*) agree more with those of undistilled oil than with those of steam distilled oil. This, therefore, clearly indicates that the sample examined by them was a mixture of volatile and fatty oils and not a volatile oil as stated by them. Again it will be seen from a comparison of data in Table I that the result of Bures and Mladkova (*loc. cit.*) are not in agreement with either

those of Crossley and Le Sueur (*loc. cit.*) or ours. Further their own data of the two different samples of the oil of *Nigella sativa* differ in their saponification, iodine and acid values.

A detailed chemical examination of the oil has shown that it consists of the glycerides of oleic (44·45%), linolic (35·99%), myristic (0·26%), palmitic (6·81%) and stearic (2·45%) acids. The glycerides of some volatile acids, both soluble and insoluble, are also present but their amounts are very small and it is difficult to isolate and identify them. We have found that the diolefinic acid present in the oil is linolic acid and not telfairic acid as reported by Bures and Mlodkova (*loc. cit.*). The two acids are isomers and thus an important result that has emerged out of this work is that the nature of soil and climate can give rise to isomeric substances in plants grown at two different places.

Experimental

About seven pounds of the authentic seeds obtained from the Punjab Ayurvedic Pharmacy, Amritsar, were dried, crushed and extracted with benzene in a five-litre flask in three successive instalments. On distilling off benzene a dark-coloured oil was obtained which on purification with animal charcoal and Fuller's earth gave a dark reddish-brown oil having a characteristic odour. This was obtained in 31% yield and deposited no sediment on keeping. It was a semi-drying oil and its physical and chemical constants are given under A in Table I.

One hundred grams of the above oil were subjected to steam distillation whereby a volatile oil (0·4%) and a pure fatty oil were obtained. The physical and chemical constants of this pure fatty oil were also determined and are given under B in Table I.

Table 1

Constants	Bures & Mladkova 1st sample	Bures & Mladkova 2nd sample	Crossley & Le Sueur	Authors' data A Oil obtained by extraction with benzene	Authors' data B Pure fatty oil obtained by steam distillation of A
Yield ...	30%	33·4 %	—	31%	—
Specific gravity.	0·8930	0·8960	0·9248(15·5°c)	0·9164(35°)	0·9152(35°)
Refractive index	—	—	—	1·4660(21°c)	1·4662(21°c)
Acid value ...	14·68	29·42	97·4	40·64	42·83
Sap. Value ...	210·6	201·98	196·4	196·9	199·6
Iodine Value ...	110·9	107·4	116·2	116·9	117·6
Acetyl Value ...	23·92	23·89	—	24·3	24·1
Unsaponifiable matter...	—	—	—	0·04	0·03
Hehner value ...	89·22	89·25	88·8	89·2	89·6
R. M. value ...	3·378	3·379	5·4	4·1	3·9

Five hundred grams of the steam distilled oil was then saponified in the usual manner by alcoholic sodium hydroxide solution. The alcohol was distilled off and

the soap solution was extracted with ether to remove the unsaponifiable matter. The soap solution was then decomposed with dilute sulphuric acid in presence of ether and the fatty acids so liberated were obtained in a solution of ether. This was washed with water till it was free from acid, dehydrated and on removal of the solvent, the fatty acids (430 gms.) were obtained having the following constants (Table 2):—

TABLE 2

Consistency ...	... Liquid (deposited no solid on keeping)
Neutralisation value ...	... 201·2
Saponification value ...	... 201·0
Mean molecular weight	... 278·9
Iodine value ...	... 119·1

As the neutralisation value and the saponification values are same, it shows that the fatty acids do not exist as anhydrides or lactones.

The R. M. value of the oil indicates the presence of some volatile acids. The neutralisation value of the insoluble volatile acids was determined by steam distillation of a known weight of insoluble fatty acids. About 700 c.c. of distillate was collected and titrated with an aqueous solution of caustic potash. It was found that 0·835 milligrams of caustic potash was required to neutralise the volatile acids from one gram of the insoluble fatty acids.

In order to find the neutralisation value of the soluble fatty acids the saponification value of the oil was found as usual. The acids were then liberated, filtered and washed as in the determination of Hehner value. These were then dissolved in neutral alcohol and titrated against a solution of potassium hydroxide. This gave the neutralisation value. The difference between the two per gram is the neutralisation value of soluble fatty acids. This was found to be 1·3.

The mixture of acids was then separated into solid and liquid acids by Twitchell's[15] lead salt alcohol process. The following table gives the percentage, iodine value, neutralisation value and the mean molecular weight of the solid and liquid acids (Table 3):—

TABLE 3

Acids.	Percentage	Iodine value	Neutralisation value	Mean molecular weight
Solid	10·11	4·6	215·4	260·5
Liquid	89·89	131·7	199·6	281·1

EXAMINATION OF THE LIQUID ACIDS

Elaidin test—The liquid acids gave a positive elaidin test showing the presence of oleic acid.

Oxidation with potassium permanganate—Ten grams of the acids were dissolved in aqueous caustic potash and oxidised by a dilute solution of potassium permanganate at room temperature according to the method of Hazura,[16] whereby a dihydroxy stearic acid M.P. 131°c and a tetrahydroxy stearic acid M.P. 172°c were obtained showing the presence of oleic and linolic acids. Hexahydroxy stearic acid was not produced at all showing the complete absence of linolenic acid.

Bromination method—The constituents of the liquid acids were determined quantitatively by the method of Eibner and Muggenthalor,[17] modified by Jamieson and Boughmann[18] wherein the bromine addition products of the acids were prepared in ether at —10°c and examined. The results are given below (Table 4):—

TABLE 4

Weight of acids taken	5·7533 gms.
Weight of tetrabromide	3·6372 gms.
Melting-point of tetrabromide	113°c
Weight of linolic acid	2·4340 gms
Weight of residue (di- and tetrabromide)	6·8551 gms.
Percentage of bromine in the residue	40·87
Weight of tetrabromide in the residue	1·8750 gms.
Weight of dibromide in the residue	4·9801 gms.
Weight of linolic acid from residue	0·1392 gms.
Weight of oleic acid from residue	3·1780 gms.
Total weight of linolic acid...	2·5732 gms.

The following table gives the percentage of linolic and oleic acids in the liquid acids, in mixed acids and in the oil (Table 5):—

TABLE 5

Acid	Percentage in liquid acids	Percentage in mixed acids	Percentage in oil
Linolic	44·74	40·22	35·99
Oleic	55·26	49·67	44·45

The theoretical iodine value of a mixture of linolic acid (44·74%) and oleic acid (55·26%) is 130·3 and the mean molecular weight 280·9, which agree fairly well with the iodine value and the mean molecular weight of liquid acids (Table 3).

It is clear that the diolefinic acid present in the oil is linolic and not telfairic as reported by Bures and Mlodkova (*loc. cit.*). The M.P. of tetrabromide of telfairic acid is 57-58°c and that of linolic acid 113-114°c. Since the M.P. of tetrabromide isolated from liquid acids is 113°c (Table 4), this definitely confirms that the acid is linolic. Further the tetrahydroxy stearic acid isolated from the oxidation product of liquid acids melts at 172°. It is, therefore, identical with that obtained from the oxidation of linolic acid. (M.P. 173-174°) and differs from that obtained by oxidation of telfairic acid (m.p. 177°c.)

EXAMINATION OF THE SOLID ACIDS

The solid acids were first freed from traces of liquid acids by rubbing them over a porous plate. They were found to melt at 54-55°c. The acids (40 gms.) were converted to methyl esters in the usual manner and the esters after purification were subjected to fractional distillation under reduced pressure, the results of which are recorded in Table 6.

TABLE 6

Weight of esters taken—38. 9 gms.

Fraction No.	Boiling range.	Pressure.	Weight in gms.
1	Below 170°	11. 5 m.m.	2·82
2	170°—175°	11. 5 m.m.	11·46
3	175°—180°	11. 5 m m.	13·84
4	180°—185°	11. 5 m.m.	4·90
5	185°—190°	11. 5 m.m	2·98
6 (decomposed residue)	Above 190°c	11. 5 m.m.	2·61
	Total		38·61 gms.
Loss during distillation			0·29 gms.

The saponification value, the mean molecular weight and the iodine value of all the fractions were determined and the amounts of various acids in fractions were calculated according to the method of Jamieson and Boughmann[19] as shown in Table 7:—

TABLE 7

Fraction No.	Iod. value	Sap. value	M.M.W.	Myristic acid		Palmitic acid		Stearic acid		Arachidic acid		Unsaturated acids	
				gms.	%	gms.	%	gms.	%	gms.	%	gms.	%
1	1·1	215·7	260·1	0·9745	34·56	1·6690	59·20	—	—	—	—	0·0236	0·84
2	1·9	205·9	272·5	—	—	9·4360	86·70	0·7681	6·70	—	—	0·1655	1·44
3	2·7	199·5	281·1	—	—	7·9790	57·63	4·9940	36·08	—	—	0·2840	2·05
4	3·8	201·2	279·8	—	—	3·2010	65·33	1·3100	26·72	—	—	0·1415	2·89
5	4·8	186·2	300·7	—	—	—	—	2·1180	81·17	0·2746	10·52	0·0952	3·65
TOTAL				0·9745		22·7850		9·1901		0·2746		0·7098	

Fraction No. 1—The molecular weight of the fraction shows that it is a mixture of methyl esters of myristic and palmitic acids. The mixture of acids on liberation from the esters melted at 55°c. This was repeatedly fractionally crystallised from acetone and two fractions were obtained, one melting at 62-63°c and the other at 49-50°c. The higher melting fraction was palmitic acid as its melting-point was not

depressed by the addition of pure palmitic acid. The lower melting fraction was myristic acid and its melting-point was not depressed by the addition of a genuine sample of myristic acid. The percentage of palmitic and myristic acids in this fraction are 59·2 and 34·56 (Table 7).

Fraction No. 2—The molecular weight of the fraction shows that it is almost pure methyl palmitate. The acid liberated from the fraction on repeated crystallisation from acetone proved to be palmitic acid M.P. 61-62°c; the melting-point was not depressed by the addition of a genuine sample of palmitic acid. The composition of this fraction corresponds to 86·7 % palmitic and 6·7 % stearic acid (Table 7).

Fraction No. 3—The molecular weight lies between that of methyl palmitate and methyl stearate. The free acids on liberation and repeated fractional crystallisation from acetone gave two products, one melting at 67-68°c (stearic) and the other at 62-63°c (palmitic). These melting-points were not depressed by the addition of pure stearic and palmitic acids respectively. The fraction contains 57·63 % palmitic and 36·08 % stearic acids (Table 7).

Fraction No. 4—This was identical with fraction No. 3 and gave on liberation and purification stearic and palmitic acids. The results were confirmed by the determination of mixed melting-points. The fraction contains 65·34 % palmitic and 26·42 % stearic acid (Table 7).

Fraction No. 5—The molecular weight shows that it is almost pure methyl stearate with a minute quantity of the methyl ester of a higher fatty acid. The liberated acid on crystallisation from acetone gave stearic acid M.P. 67-68°c which was not depressed by the additions of pure stearic acid. The fraction contains 81·17 % stearic acid (Table 7).

The percentages of various solid acids in mixed solid acids, in mixed acids and in oil are given below (Table 8):—

TABLE 8

Acid	Percentage in solid acids	Percentage in mixed acids	Percentage in oil
Myristic.	2·88	0·291	0·26
Palmitic.	67·22	6·795	6·31
Stearic.	27·11	2·74	2·45

EXAMINATION OF THE UNSAPONIFIABLE MATTER

The unsaponifiable matter obtained gives the colour reactions for a phytosterol, sitosterol found in most vegetable oils, but the amount being too small it could not be fully investigated.

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A NOTE ON THE OCCURRENCE OF *ENTEROMORPHA* AT ALLAHABAD

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Most of the members of Ulvaceæ are marine but *Enteromorpha* and *Monostroma* may be found in fresh water. The former is capable of growing under wide range of salinity. It is, therefore, found ascending some distance into estuaries. *Enteromorpha compressa* has also been reported from cliffs.

In India *Enteromorpha* has not been reported from fresh water, but in April, 1937, the author found it growing abundantly in the Jamuna river. The thallus was attached to the bottom of the river and it was growing mixed up with a species of *Hydodictyon*. Some material was sent to Profs. M. O. P. Iyenger and G. M. Smith for confirmation and naming of the species. They agree with generic determination but the full name is not given so far.

The occurrence of *Enteromorpha* at Allahabad is interesting because there is no cliff in the neighbourhood and the place is far removed from the sea. A detailed account will appear later.

STUDIES ON THE SIX NEW SPECIES OF THE GENUS *NEODIPLOSTOMUM*, RAILLIET, 1919 (FAMILY *DIPLOSTOMIDAE* POIRIER, 1886). PART I—NEW SPECIES OF THE SUBGENUS *NEODIPLOSTOMUM* DUBOIS, 1937.

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Communicated by Dr. H. R. Mehra

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Of the six species described in this paper two *N. hawkei*, n. sp. and *N. nisus* n. sp. were obtained from the hawks, *Accipetres nisus malanoschistus*, two *N. muthiari* n. sp. and *N. brachypteris* n. sp. were obtained from woodpecker *Brachypternus bengalensis bengalensis*, one *N. rufeni* n. sp. from the kite, *Buteo rufinus rufinus* and one *N. eudynamis* n sp. from the Indian koil *Eudynamis scolopaceous* caught from the villages Muthiara and Sulemsarai near Allahabad. So far the species of this genus *Neodiplostomum* from India the accounts of which are published are *N. tytense* Patwardhan, 1935 from *Tytoalba stertens*, *N. globiferum* Verma, 1936 from *Cuckoo*, *N. mehranium* Vidyarthi, 1938 from *Haliaetus leucroyphus* and *N. laruei* Vidyarthi, 1938 from *Sarcogyps calvus*, but whether they belong to the subgenus *Neodiplostomum* or *Conodiplostomum* cannot be determined at present as the presence or absence of the genital cone is not mentioned in their accounts. The account of the three species of the genus *Neodiplostomum* described by Gupta (1937) in his M. Sc. thesis which is not yet published are *N. duboisii* n. sp. Gupta (1937) from Owl, *Ketupa sclonensis handwickii*, *N. austerense* n. sp. Gupta (1937) from hawk *Austur bandius lencomalanara*, and *N. hieractii* n. sp. Gupta (1937) from *Hieraitus fasciatus fasciatus*.

The genus *Neodiplostomum* was created by Railliet, 1919 to supersede the genus *Diplostomum*, Brandes 1888, for certain species, which had been included under the latter genus, specially *N. spathulae-forme* (Brandes, 1888). Brandes, (1888) restricted the genus *Diplostomum* V. Nordman, 1932 for the *diplostomes* of reptiles and three *diplostomes* of Birds, *i.e.*, *Diplostomum grande* (Diesing 1850) Dubois, 1936; *Diplostomum spathula* Creplin, 1829; *Diplostomum spathulae-forme*, Brandes. Railliet in 1919 separated the last three species under his new genus *Neodiplostomum*. Poirier in 1886 created the family *Diplostomidae* for these three species as well as the five species parasitic in reptiles whose metacercarial stage was found encysted in the eyes of various fishes. Dubois in 1936 separated the *Diplostome* species of reptiles under his new family *Proterodiplostomidae* and retained the family *Diplostomidae* Poirier for the *Diplostome* parasites of birds and mammals. The family name *Diplostomidae*. Poirier, 1886 has priority over *Allariidae* Tubangui, 1922. The

subfamily *Diplostominae* to which the genus *Neodiplostomum* belongs was created by Monticelli in 1888 (May—August) having priority over its homoname *Diplostominae* Brandes, 1888 (November).

Dubois (1937) after examining a large amount of material in about 300 preparation of the genus *Neodiplostomum* came to the conclusion that the genus should be divided into two subgenera on the basis of the presence or absence of the genital cone in the bursa copulatrix. The subgenus *Neodiplostomum*, Dubois, 1937 stands for the species in which the genital cone is absent and the anterior testis is asymmetrical, whereas the subgenus *Conodiplostomum* Dubois contains species in which the hermaphroditic canal opens into the bursa copulatrix after traversing through the genital cone, and in which both the testes are as a rule symmetrical. Dubois has designated the type of the genus, *Neodiplostomum spathulae-forme* (Brandes, 1888) as the type species of the subgenus *Neodiplostomum* and *Neodiplostomum spathula* (creplin 1829) La Rue, 1926 as the type of the subgenus *Conodiplostomum* Dubois.

My best thanks are due to Dr. H. R. Mehra under whose guidance I had the privilege to carry on this work. I am very grateful to him for his keen interest, valuable advice, criticisms, and help in consulting literature from his personal library. I also thank Dr. D. R. Bhattacharya for the laboratory facilities offered to me and for his interest in the progress of this work.

Neodiplostomum brachypteris n. sp.

This species was obtained from the small intestine of three woodpecker *Brachypternus bengalensis bengalensis* caught from Sulemsarai a village near Allahabad. Out of the six hosts examined only three yielded the parasites on the average 10 each. When examined in the living condition in the physiological salt solution they were white in colour and showed a constant elongation and contraction of the anterior body. Sexually mature worm 1·088—1·209 m.m. in length and 0·561—0·714 m.m. in maximum breadth in the region of the holdfast organ, is distinctly divided into the longer ellipsoid anterior body and a short cylindrical posterior body. The anterior body armed with minute backwardly directed cuticular spines measures 0·595—0·799 m.m. in length and 0·561—0·714 m.m. in breadth. The cylindrical hindbody measures 0·357—0·544 m.m. in length and 0·306—0·374 m.m. in maximum breadth in the region of the anterior testis. The average ratio in the length of the forebody and hindbody is approximately 3 : 2. Oral sucker 0·068—0·085 m.m. in length and 0·063—0·085 m.m. in breadth is almost rounded and subterminal. The ventral sucker almost equal in size to the oral sucker is slightly broader than long lies 0·27—0·338 m.m. distance behind the anterior end, and measures 0·065—0·084 m.m. in length and 0·087—0·1 m.m in breadth. Average ratio in the size of the oral sucker and ventral sucker is nearly 3 : 4. The holdfast organ

0·1—0·125 m.m. in length and 0·063—0·088 m m. in breadth lies 0·075—0·088 m.m. behind the ventral sucker. Prepharynx is absent. The pharynx measures 0·05—0·063 m.m. in length and 0·043—0·058 m.m. in breadth. The oesophagus is very small practically absent. The intestinal bifurcation lies at 0·153 m.m. distance from the anterior extremity. The caeca in the forebody run on either side of the holdfast organ and the ventral sucker 0·125, and 0·075 m.m. away from them respectively. In the hindbody they are confined to the ventrolateral margins, terminating near the hinder end.

Testes tandem, postovarion, lie close behind one another in the anterior two-third length of the hindbody. The anterior testis is asymmetrical and pear-shaped with the pointed end extending a little to the right side of the median line and the broad end pressed against the left body margin measuring 0·088—0·138 m m. in

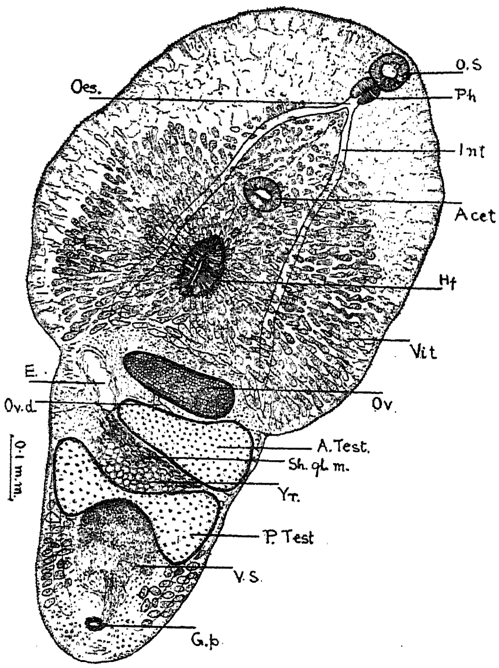

Fig. 1.—*Neodiplostomum brechipteris* n. sp.

Acet, Acetabulum; Ec., Excretory Canal; E., Egg; Ga., Genital atrium; Gp., Genital pore; Gpa., Genital papilla; Gc., Genital cone; Hf., Holdfast organ; Int., Intestinal caecum; L.C. Laurer's Canal; O.S.. Oral sucker; Oes., Oesophagus; Ov., Ovary; Ovd., Oviduct; Ph., Pharynx; Sh. gl. m., Shell gland mass; R.S., Recepticulum seminalis; Test., Testis; A.Test., Anterior testis; P. Test, Posterior testis; U. uterus; V.A., Vasa afferencia; V.S., Vesicula seminalis; A.V.S., Anterior vesicula seminalis; P.V.S., Posterior vesicula seminalis; Vit., Vitellaria; Yr., Yolk reservoir.

length and 0·163—0·289 m.m. in breadth. The symmetrical, dumbel-shaped posterior testis is much constricted in the middle and occupies almost the entire width of the

hindbody measuring 0·100—1·113 m.m in length and 0·238—0·278 m.m. in breadth. The ovary transversely elongated of somewhat oval form measures 0·053—0·093 m.m. in length and 0·183—0·215 m.m. in breadth and lies just at the junction of the forebody and hindbody at 0·561 m.m. behind the anterior extremity. The oviduct arises from the middle of the posterior margin of the ovary and runs posteriorly to the left side of the anterior testis. The large yolk reservoir lies between the two testis overlapping the shellgland mass. The uterus containing a few ova, 4—6 in number, runs forward upto the anterior margin of the ovary and then turns backward as a straight tube in the median line towards the genital atrium. The ova measures 0·1—0·117 m.m. in length and 0·062·m.m. in breadth. The small slightlyc onvoluted vesicula seminalis lies just behind the posterior testis. The small ductus ejaculatorius unites with the terminal end of the uterus to form the small hermaphroditic canal opening directly into the genital atrium, which lies dorsally at 0·045—0·050 m.m. in front of the hinder end. The genital cone is absent. The numerous pear-shaped vitelline follicles are strongly developed in the anterior body surrounding the holdfast organ extending from the bifurcal Zone to almost the posterior end of the body as far as the genital opening. In the hind-body they are laterally situated overlapping the intestinal coeca.

Discussion.—This species belongs to the subgenus *Neodiplostomum* Dubois on account of the absence of the genital cone. The bursa copulatrix of the type species *N. Spathulae forme* (Brandes, 1888) Railliet 1919 parasitic in Strigis has not been described. The species of the subgenus *Neodiplostomum* are *N. Conicum* Dubois 1937, *N. travassosi* Dubois 1937, and *N. cochlear* (Krasuse 1914) La. Rue 1926, all parasitic in Strigis, *N. microcotyle* Dubois, 1937, *N. spathoides* Dubois, 1937, *N. obscurum* Dubois, 1937, *N. inaequipartitum* Dubois, 1937, *N. paraspathula* Noble, 1936, *N. pseudattenuatum* (Dubois, 1928), Dubois, 1932 and *N. biovatum* Dubois, 1937 parasitic in Accipetres, *N. ellipticum* (Brandes, 1888) La. Rue, 1926 parasitic in Cuculi, *N. rhamphasti* Dubois, 1937 in Pici. and N. lucidum La. Rue and Bosma, 1927 parasitic in Marsupials. *N. brachypteris* n. sp. resembles closely *N. rhamphasti* Dubois on account of the absence of the prepharynx, small size of the oesophagus and its habitat in Pici. But it differs from it in shape and size of the body, in relative length and breadth of the anterior body and posterior body. The anterior body in *N. rhamphasti* Dubois is lanceolate with the posterior part behind the holdfast organ large and rounded, and is about twice the length of the short conical posterior body, while in the new species it is somewhat oval only a little longer than the posterior body. The cuticular spines which are present in the new species have not been mentioned in *N. rhamphasti*. Only a few ova are present in *N. brachypteris* n. sp. while in *N. rhamphasti* they are numerous. *N. conicum* Dubois, 1937, and *N. cochlear* (Krasuse 1914) La. Rue, 1926 parasitic in Strigis are easily separated from our species on account of their asymmetrical testes, shape and size of the body, shape and relative lengths of the fore-body and hind-body, profuse development of

the vitellaria in the hind-body and the hosts belonging to different orders of birds. *N. travassosi* Dubois, 1937, parasitic in Strigis, which resembles the new species in the asymmetrical shape of the anterior testis, stands also quite apart on account of its anterior oblong, lanceolate body which is of nearly twice the length of the posterior sub-cylindrical body, in the position, shape and size of the holdfast organ, shape of the posterior testis and extension of the vitellarian follicles on the ventral surface of the entire posterior body. The other species of the subgenus which differs in many features are not closely related to *N. brachypteris* n. sp.

Host.—*Brachypternus bengalensis bengalensis.*

Habitat.—Small intestine.

Locality.—Allahabad (U. P., India).

Neodiplostomum hawkei n. sp.

14 specimens were collected on 16th November, 1940, from the small intestine of Indian hawks *Accipetres nisus malanoschistus* caught from the village Muthiara near Allahabad. In all 6 hosts were examined and 2 of them were found to be infected, one with 6 and the other with 8 specimens of this diastomes. Body 1·360—1·479 m.m. long. Forebody 0·714—0·850 m.m. long and 0·816—1·037 m.m. broad, spherical and armed with minute backwardly directed cuticular spines. Hindbody cylindrical 0·561—0·697 m.m. long and 0·459—0·697 m.m. broad. Average ratio in length of forebody and hindbody nearly 6:5. Oral sucker Subterminal, oval, 0·034—0·075 m.m. long and 0·055—0·077 m.m. broad. Ventral sucker 0·037—0·068 m.m. long and 0·068—0·119 m.m. broad, transversely oval at 0·187—0·262 m m. distance behind anterior end and 0·125 m m. in front of holdfast organ. Average ratio in size of oral and ventral sucker nearly 3:4. Holdfast organ 0·187—0·238 m.m. long and 0·068—0·140 m.m. broad at 0·323—0·425 m.m. behind anterior end. Prepharynx absent; muscular pharynx 0·051—0·068 m.m. long and 0·045—0·088 m.m. broad; oesophagus very small; intestinal bifurcation 0·119—0·136 m.m. behind anterior end; intestinal coeca in forebody run laterally on either side of ventral sucker and holdfast organ, but ventrolateral in hindbody little swollen at their terminal ends.

Testes postovarian, tandem occupying almost half length of hindbody. Anterior testis 0·187—0·225 m.m. long and 0·306—0·391 m.m. broad, asymmetrical, pear-shaped with its pointed end directed towards median line and broad end touching left body margin. Posterior testis 0·153—0·238 m.m. long and 0·425—0·595 m.m. broad transversely elongated, symmetrical occupying almost whole width of hindbody. Straight simple vesicula seminalis just behind posterior testis; narrow ductus ejaculatorius unites terminally with uterus to form small hermaphroditic canal which opens directly in genital atrium. Genital cone absent. Genital opening lies dorsally at 0·043 m.m. in front of hinder end. Ovary 0·083—0·119 m.m. long and 0·187—0·272

m.m. broad, transversely elongated, situated a little in front of body constriction at 0·714 m.m. from anterior end. Oviduct arises from anterodorsal side of ovary, proceeds backward by right side of anterior testis to intertesticular shellgland complex, yolk reservoir dorsal to shell gland mass, pear-shaped with pointed end towards median line and broad end close to right body margin measures 0·068 m.m. in length and 0·306 m.m. in breadth. Ova 0·102—0·119 m.m. long and 0·034—0·06 m.m. broad 6 to 12 in number and operculated. Vitalline follicles numerous pear-shaped profusely developed in forebody poorly developed in hindbody commences just in front of ventral sucker and terminates near hinder extremity of body. Ventrolateral with slight extension towards median line in hindbody.

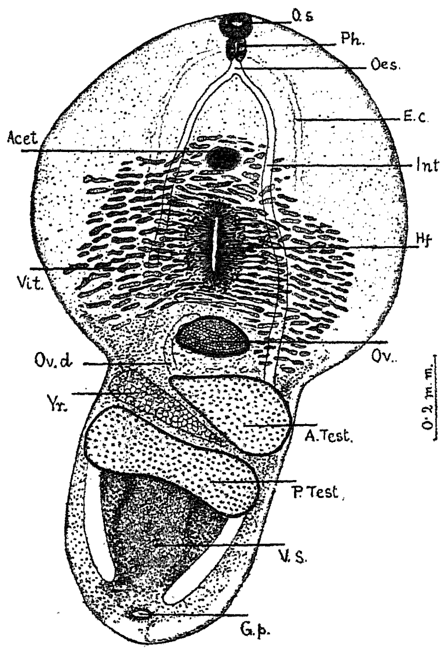

Fig. 2.—*Neodiplostomum hawkei* n. sp.
(Lettering as in Fig. 1)

Discussion.—This species comes under the subgenus *Neodiplostomum* Dubois on account of the absence of the genital cone in the bursa copulatrix. Of all the species known under this subgenus it stands close to *N. pseudattenuatum* (Dubois, 1928) Dubois, 1932 *N. laruei* Vidyarthi 1938 on account of the absence of the prepharynx, presence of a short oesophagus, more or less equal size of the suckers, the asymmetrical form of anterior testis, vitellaria commencing in front of ventral sucker

and the habitat in the host birds Accipetres. It differs from *N. pseudattenuatum* in the shape and size of the body, shape of the forebody, measurements of the anterior body, in the shape of the posterior testis (posterior testis horse-shoe shaped and ventrally concave in *N. pseudattenuatum*) and in the position of the ovary which in the new species is little in front of the hindbody. In *N. pseudattenuatum* the vitelline follicles are strongly condensed near the end behind the testes unlike that in *N. hawkei* n. sp. *N. laruei* Vidyarthi differs from the new species in the shape and large size of its body, in the shape and size of the anterior testis and shape, size and position of the ovary. Moreover, in *N. laruei* Vidyarthi almost all the organs are larger in size. *N. hawkei* n. sp. resembles *N. brachypteris* n. sp. in the absence of the prepharynx, short size of the oesophagus, equal size of the suckers asymmetrical condition of the anterior testis and absence of the genital cone, but differs from it in the shape and size of the body, in the relative measurements of the anterior and posterior bodies, in having a much larger yolk reservoir, in the shape and size of the posterior testis, in the extension of the vitellaria and on account of the hosts belonging to different orders of birds (Accipetres in the case of *N. hawkei* n. sp. and Pici. for *N. brachypteris* n. sp.)

Host.—*Accipetres nisus malanoschistus.*

Habitat.—Small intestine.

Locality.—Allahabad (U. P., India).

Neodiplostomum nisus n. sp.

In November, 1940, 3 out of 6 common Indian hawks, *Accipetres nisus malanoschistus* caught from the village Muthiara near Allahabad yielded 12 specimens of this species. Two hosts harboured three and one host six parasites in their small intestine. Body 1·564—1·87 m.m. long and 0·578—0·748 m.m. in maximum breadth in the region of the holdfast organ. Forebody 0·782—0·935 m.m. long and 0·578—0·748 m.m. broad, oval, presents a slight concavity on ventral surface and unarmed with cuticular spines. Hindbody 0·697—1·071 m.m. long and 0·374—0·476 m.m. in maximum breadth in region of posterior testis. Average ratio in length of forebody and hindbody nearly 1 : 1. Oral sucker subterminal 0·06—0·073 m.m. long and 0·045—0·057 m.m. broad. Ventral sucker 0·063—0·073 m.m. long and 0·075—0·088 m.m. broad, transversely oval, at 0·345—0·42 m.m. behind anterior end. Average ratio in size of oral and ventral suckers nearly 2 : 3. Holdfast organ 0·1—0·15 m.m. long and 0·04—0·113 m.m. broad, longitudinally oval, at 0·15—0·163 m.m. distance behind ventral sucker. Prepharynx absent; pharynx 0·035—0·058 m.m. long and 0·025—0·038 m.m. broad; oesophagus 0·06 m.m. long as measured in living specimen, hardly seen in entire mounts, intestinal bifurcation much in front of ventral sucker,

coeca in forebody lateral on either side of ventral sucker and holdfast organ, ventro-lateral in hindbody with somewhat dilated hinder ends.

Testes post ovarian, tandem occupying about half length and entire width of hind body. Anterior testis 0·153—0·238 m.m. long and 0·272—0·408 m.m. broad, asymmetrical pear-shaped with its pointed end almost touching right body margin and broad end pressed against left body wall. Posterior testis 0·153—0·225 m.m. long and 0·289—0·442 m m. boad, symmetrical, with slight constriction in the middle, at 0·136 m.m. behind anterior testis. Vesicula seminalis well developed, simple, just behind posterior testis. Genital atrium spaceous, situated dorsally a little in front of hinder end; a small muscular papilla which may be mistaken for genital cone, present just in front of genital atrium. Hermaphroditic canal opens in the genital atrium just behind muscular papilla. Genital cone absent. Ovary 0·102—0·17 m.m. long and 0·187—0·217 m.m broad transversely elongated, situated just at junction of forebody and hindbody at 0·731—0·762 m.m. from anterior end. Oviduct arises from posteroventral side of ovary. Shell gland mass intesterticular

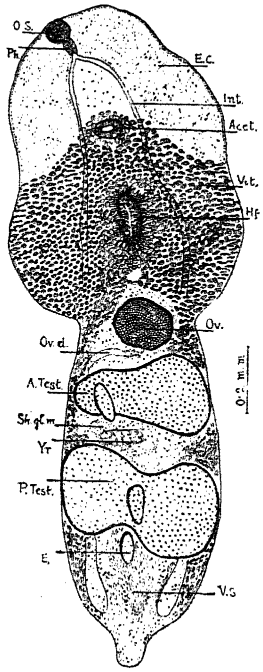

Fig. 3.—*Neodiplostomum nisus* n. sp.
(Lettering as in Fig. 1)

shifted more towards right side, yolk reservoir dorsal to shell gland mass. Laurer's canal opens dorsally anteriorly to posterior testis near right body margin. Ova

large 10—12 in number 0·105—0·110 m.m. long and 0·05—0·052 m.m. broad. Vitellaria composed of numerous small rounded follicles, profusely developed in forebody, poorly developed, confined to ventrolateral margins in hindbody extends from just in front of ventral sucker to posterior end of intestinal coeca.

Discussion.—*N. nisus* n. sp. though it belongs to the subgenus *Neodiplostomum* on account of the absence of the genital cone differs from all the species of this subgenus on account of the presence of a genital papilla situated in front of the opening of the hermaphroditic canal at the base of the genital atrium. It closely resembles *N. hawkei* n. sp. on account of the absence of a prepharynx, small size of the oesophagus, nearly equal size of the suckers, asymmetrical shape of the anterior testis, commencement of the vitellaria just in front of the ventral sucker and habitat in the same host, but it can be easily differentiated from it on account of the different size and shape of body, shape and relative sizes of the forebody and hindbody, large size and shape of both the testis, shape, size and position of the ovary, small size of the yolk reservoir and large size of the ova.

Host.—*Accipetres nisus malanoschistus.*

Habitat.—Small intestine.

Locality.—Allahabad (U. P., India).

(Bibliography is given at the end of Part II.)

STUDIES ON THE SIX NEW SPECIES OF THE GENUS *NEODIPLOSTOMUM*, RAILLIET, 1919 (FAMILY *DIPLOSTOMIDÆ* POIRIER, 1886)—PART II.—NEW SPECIES OF THE SUBGENUS *CONODIPLOSTOMUM* DUBOIS, 1937

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Neodiplostomum rufeni n. sp.

Thirty specimens of this distomes were collected from the small intestine of the kite *Buteo rufinus rufinus* obtained from Sulemsarai, a village near Allahabad, 3 out of 5 kites were infected, each harbouring 8—12 specimens. Body, 1·428—2·125 m.m. long and 0·663—0·986 m.m. maximum breadth in the region of holdfast organ. Anterior body 0·527—0·850 m.m. long and 0·663—0·986 m.m. broad, more or less rounded with a hollow concavity on the ventral surface and armed with minute backwardly directed cuticular spines. Hindbody 0·629—1·275 m.m long and 0·459 - 0·748 m.m. broad, cylindrical. Average ratio in length of forebody and hindbody nearly 4 : 5. Oral sucker almost terminal 0·05—0·07 m.m. long and 0·060—0·075 m.m. broad. Ventral sucker 0·045—0·063 m.m. long and 0·0675 – 0·0925 m.m. broad, at 0·195—0·350 m.m. distance behind anterior end. Average ratio in size of oral and ventral suckers nearly 1 : 1. Holdfast organ, 0·1125—0·1500 m.m. long and 0·045—0·070 m.m. broad, at 0·055—0·07 m.m distance behind ventral sucker. Prepharynx absent. Pharynx muscular 0·04—0·062 m.m. long and 0·0425—0·0625 m.m. broad Oesophagus 0·025 m.m. long as measured in living specimens, hardly seen in entire mounts. Intestinal bifurcation at 0·1525 m.m. from anterior end and 0·1325 m.m. in front of ventral sucker, intestinal coeca in forebody lateral on either side of ventral sucker and holdfast organ at 0·050 m m and 0·125 m.m. away from them respectively, but ventrolateral in hindbody.

Testes, post ovarian, tandem, occupying almost entire width of hindbody. Anterior testis 0·150—0·306 m.m. long and 0·493—0·668 m.m. broad, asymmetrical and pear-shaped with the pointed end directed towards the left and broad end towards the right side. Posterior testis 0·204—0·374 m.m. long and 0·408—0·714 m.m. broad, symmetrical and somewhat dumbel-shaped. Vesicula seminalis straight, situated behind posterior testis. Hermaphroditic canal passes through a well-developed muscular genital cone. Ovary 0·085—0·170 m.m. long and 0·340—0·391 m.m. broad transversely oval and situated at the commencement of hindbody at 0·646—0·884 m.m. behind the

anterior extremity. Oviduct arises from anterodorsal side of ovary and proceeds backward on left side to intertesticular shellgland complex. Ascending uterus runs to right side of anterior testis upto the junction of forebody and hindbody where bends to come down as straight median tube. Genital cone large 0·095—0·100 m m. long and 0·0825—0·0875 m.m. broad, filling almost entire genital atrium. Genital opening dorsal and 0·0675 m.m. in front of the hinder end. Vitellaria composed of numerous small, rounded follicles, profusely developed both in forebody and hindbody, extending from about halfway between intestinal bifurcation and ventral sucker up to hinder extremity of body. On account of the profuse development they overlap the internal organs which are thus not properly seen in entire mounts but can be studied in sections. Ova numerous, 0·0875—0·095 m.m. long and 0·045—0·0525 m.m. broad.

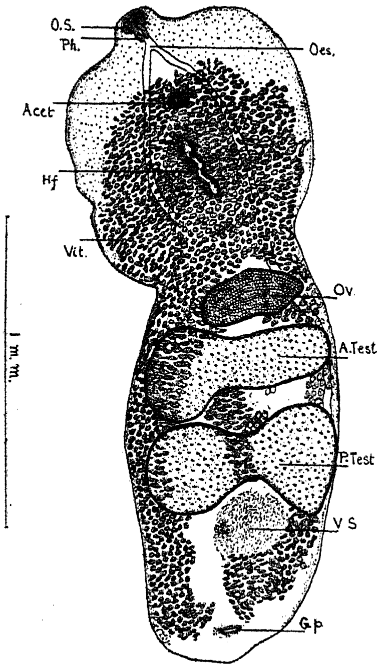

Fig. 1.—*Neodiplostomum rufeni* n. sp.

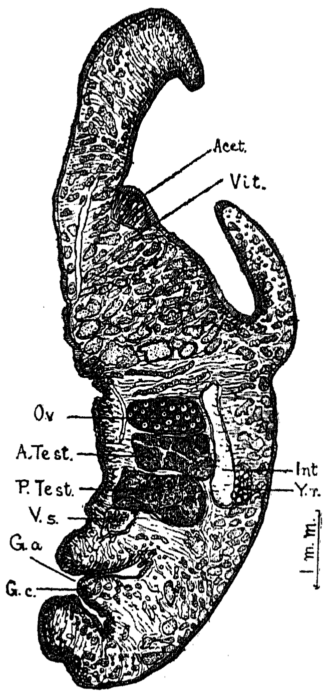

Fig. 2.—*L. S. of Neodiplostomum rufeni* n. sp.

Acet., Acetabulum; E.C., Excretory Canal; E., Egg; Ga., Genital atrium; Gp., Genital pore; Gpa., Genital papilla; Gc., Genital cone; Hf., Holdfast organ; Int., Intestinal caecum; L.C., Laurer's canal; O.S., Oral sucker; Oes., Oesophagus; Ov., Ovary; Ovd., Oviduct; Ph., Pharynx; Sh. gl. m., Shell gland mass; R.S., Recepticulum seminalis; Test., Testis; A. Test., Anterior testis; P. Test., Posterior testis; U., uterus; V.A., Vasa afferencia; V.S., Vesicula seminalis; A.V.S., Anterior vesicula seminalis; P.V.S., Posterior vesicula seminalis; Vit., Vitellaria; Yr., Yolk reservoir.

Discussion.—According to the classification given by Dubois in 1937 this species comes to the subgenus *Conodiplostomum* on account of the presence of a genital cone. This subgenus contains so far *N. australiense* Dubois, 1937, *N. acutum* Dubois, 1937, *N. spathula* (creplin, 1829) La Rue, 1926, *N. sarcorhamphi* Dubois, 1937, *N. palumbarii* Dubois, 1937 and *N. perlatum* Ciurea, 1929 all parasitic in Accipetres and *N. brachyurum* (Nicoll, 1914) Dubois, 1937 parasitic in Strigis. *N. brachyurum* can be easily separated from the new species on account of its different host specificity. It can further be differentiated on account of differences in the shape and size of the forebody and hindbody and the size and shape of the testes. Both the testes are symmetrical in *N. brachyurum*, whereas in *N. rufeni* n. sp. the anterior testis is asymmetrical and posterior testis symmetrical. The new species differs from *N. australiense* in general shape and size of the body (1·53—1·65 m m. long in *N. australiense* and 1·428—2·125 m.m. in *N. rufeni* n. sp.), in the shape and relative size of the forebody to the hindbody. Moreover in *N. australiense* the forebody is much longer and narrower than that of new species. Oesophagus is longer in *N. australiense* than in *N. rufeni* n. sp. (oesophagus 0·05—0·09 m.m. long in *N. australiense* and 0·025 m.m. long in *N. rufeni* n. sp.). The holdfast organ also differs in shape and size and occupies a much more posterior position in *N. australiense*. *N. rufeni* n. sp. differs from *N. acutum* in shape and size of the body and in shape and size of ovary. The ovary is spherical in *N. acutum* whereas it is transversely oval in *N. rufeni* n. sp. The shape and size of the holdfast organ also differs in the two species. From *N. spathula* the type species of the subgenus, *N. palumbarii* and *N. sarcorhamphi* our species can be differentiated on account of the difference in the relative lengths of the forebody and the hindbody and in the shape of the testes. The hinder body is longer than forebody in *N. rufeni* n. sp. reverse to that in above-mentioned species. The new species resembles closely *N. perlatum* in having the anterior testis asymmetrical, which is exceptional in the subgenus *conodiplostomum*. It further resembles *N. perlatum* on account of the small sizes of the oesophagus and absence of the prepharynx, but it differs remarkably in the shape and size of the body, in the relative sizes of the forebody and hindbody, in the position, shape and size of the holdfast organ and position of the ventral sucker much further behind the intestinal bifurcation *N. mehranium* Vidyarthi, 1938 and *N. laruei* Vidyarthi, 1938 which resemble the new species on account of their being parasitic in the Accipetres differ much from the new species on account of the shape and size of the body, in the ratio of the size of the forebody to hindbody in the shape and size of the testes. *N. mehranium* further differs from it on account of the highly massive holdfast organ which is provided with well-developed adhesive glands. As Vidyarthi has not mentioned anything about the presence or absence of the genital cone in his species it is not possible to assign them to any of the two subgenera

and compared properly with the new species. Out of the three species **N. duboisii* Gupta, 1937, **N. austerense* Gupta, 1937 and **N. hieractii* Gupta, 1937 (accounts not yet published), the last two resemble *N. rufeni* n. sp. in host specificity but the former differs being parasitic in strigis. *N. rufeni* n. sp. differs from all the above species on account of the large size of the genital cone which is inconspicuous in the above-mentioned species, in the shape of the body, relative sizes of the forebody and hindbody, in the absence of the prepharynx and extremely short length of the oesophagus. The new species also differs from *N. austerense* Gupta, 1937 in the shape of the posterior testis which is symmetrical in the new species and horseshoe-shaped with the concavity directed ventrally in *N. austerense*.

Host:—*Buteo rufinus rufinus*.

Habitat:—Small intestine.

Locality:—Allahabad (U.P., India).

Neodiplostomum muthiari n. sp.

Six specimens of this species were collected from the small intestine of one out of six woodpeckers, *Brachypternus bengalensis bengalensis* caught from Muthiara, a village near Allahabad. Body length 1·581—2·091 m.m. Forebody 0·850—1·122 m.m. long and 1·003—1·173 m.m. broad, more or less rounded, flat, armed with minute backwardly directed spines. Hindbody 0·680—1·003 m.m. long and 0·578—0·680 m.m. broad, cylindrical. Average ratio in length of forebody and hindbody nearly 6 : 5. Oral sucker 0·080—0·085 m.m. long and 0·080—0·100 m.m. broad, nearly spherical and terminal. Ventral sucker, transversely elongated, 0·078—0·088 m.m. long and 0·098—0·115 m.m. broad, at 0·338—0·418 m.m. distance behind anterior end. Average ratio in size of oral and ventral suckers nearly 5 : 6 Holdfast organ 0·163—0·225 m.m long and 0·088—0·175 m.m. broad at 0·125—0·163 m.m distance behind ventral sucker. Prepharynx absent; muscular pharynx 0·068—0·075 m m. long and 0·053—0·068 m.m broad, oval; oesophagus very small, intestinal bifurcation at 0·1825 m.m. distance behind anterior extremity; intestinal coeca lateral in forebody on either side of ventral sucker and holdfast organ at 0·068—0·095 m.m. and 0·119 m.m. away from them respectively, but ventrolateral in hindbody.

Testes, postovarian, tandem separated from one another in median plane by 0·119—0·187 m.m. distance; anterior testis 0·136—0·204 m.m. long and 0·340—0·391 m.m. broad, transversely elongated, asymmetrical somewhat pear-shaped with narrow end towards median line, broad end towards left body margin, posterior testis 0·136—0·204 m.m. long and 0·408—0·459 m.m. broad, symmetrical, transversely elongated and somewhat dumbel-shaped. Vesicula seminalis close behind posterior

Gupta gave an account of *N. duboisii*, *N. austerense*, and *N. hieractii* in his thesis submitted for the M.Sc. examination in Zoology of Allahabad University.

testis, large and coiled. Genital cone as seen in one specimen is present, but as sections were not cut its position requires confirmation. Ovary 0·102—0·153 m.m. long and 0·255—0·289 m.m. broad, transversely oval, at 0·986 m.m. behind anterior end, shell gland mass and yolk reservoir intertesticular. Ova 0·092—0·110 m.m. long and 0·045 – 0·055 m.m. broad, 2—4 in number, operculated, yellowish brown in colour. Vitellaria composed of numerous small rounded follicles profusely developed in forebody, poorly developed and dorsolateral, in hindbody extends from a little behind intestinal bifurcation to hinder extremity of body.

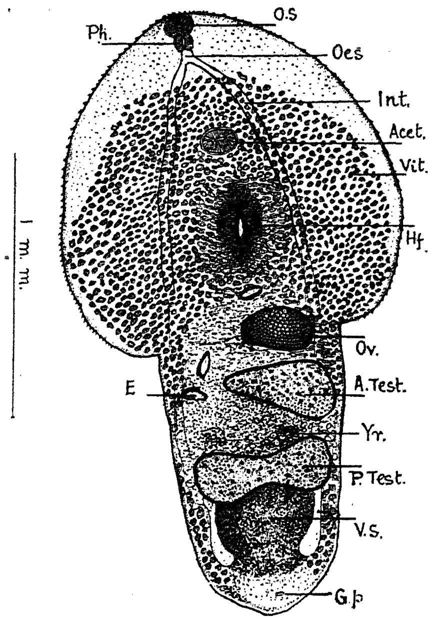

Fig. 3.—*Neodiplostomum muthiari* n. sp.
(Lettering as in Fig. 1.)

Discussion.—I assign provisionally this species to the subgenus *conodiplostomum* because of the presence of the genital cone, though seen in one toto preparation requires to be confirmed. It may further be mentioned that no other species from the host birds of the order Pici has been yet included in this subgenus. The new species approaches *N. perlatum* (Ciurea, 1919) Ciurea 1929, and *N. rufeni* n. sp. on account of the absence of the prepharynx, presence of a short oesophagus and the asymmetrical condition of the anterior testis, but it is distinguished from the former in the broader size and oval shape of the forebody; presence of cuticular spines in the forebody, in the shape of the testes, in the extension of the vitellaria (in *N. perlatum* the vitellaria commence behind ventral sucker and extend medially in the hindbody overlapping the gonads) and the host birds also belong to two different

orders. (Accipetres in the case of *N. perlatum* and Pici in the case of *N. muthiari* n. sp.). From *N. rufeni* n. sp. it can be easily separated on account of the difference in the relative length and breadth of the forebody and hindbody, the ventral sucker being larger than the oral sucker, the vitellaria not extending towards the median plane in the hindbody and the hosts belonging to different orders of birds (Accipetres in the case of *N. rufeni* n. sp. and Pici in the case of *N. muthiari* n. sp.). The anterior testis in *N. muthiari* n. sp. does not occupy the entire breadth of the hindbody as in *N. rufeni* n. sp. Though the new species resembles *N. laruei*, Vidyarthi, 1938 in the shape of the forebody and hindbody, presence of cuticular spines, absence of the prepharynx, presence of a very short oesophagus and profuse development of the vitellaria in the forebody and its restriction to the lateral regions in the hindbody, it differs from it in the size of the body, in the size and shape of the holdfast organ, size and shape of the anterior testis and ovary and the host belonging to different orders of Birds (*N. laruei* is parasitic in *Sarcogyps calvus* of the order Accipetres).

Host :—*Brachypternus bengalensis bengalensis*.

Habitat : – Small intestine.

Locality :—Allahabad (U.P., India).

Neodiplostomum eudynamis n. sp.

Twelve specimens of this distomes were collected from small intestine of the Indian Koil *Eudynamis scolopaceous*, caught from Sulemsarai, a village near Allahabad. Two birds examined yielded eight and four parasites each. About 30 specimens in toto mounts and 6 specimens for section cutting were also given to me by Dr. H. R. Mehra for study for which I am much indebted to him. Body 1·88—3·14 m.m. long and 0·935—1·190 m.m. maximum breadth in region of holdfast organ. Anterior body 0·901—1·275 m.m. long and 0·935—1·390 m.m. broad, ellipsoid armed with minute backwardly directed spines. Hindbody cylindrical 0·595—1·820 m.m. long and 0·561—0·935 in maximum breadth in region of anterior testis. Average ratio in length of forebody and hindbody nearly 1 : 1. Oral sucker subterminal, longitudinally oval 0·075—0·118 m.m. long and 0·07—0·113 m.m. broad as measured in toto mounts and 0·088 by 0·075 m.m. in sections. Ventral sucker transversely oval 0·038—0·068 m.m long and 0·113—0·113 m.m. broad situated at 0·352—0·660 m.m. distance behind anterior end and midway between intestinal bifurcation and holdfast organ. Average ratio in sizes of oral and ventral suckers nearly 5 : 4. Holdfast organ 0·195 – 0·272 m.m. long and 0·112—0·153 m.m. broad, elongated oval, at 0·125 m.m. distance behind ventral sucker. Prepharynx absent, pharynx muscular 0·075—0·113 m.m. long, 0·065—0·113 m.m. broad in pressed specimens and 0·062—0·088 m.m. long and 0·050 – 0·070 m.m. broad in sections; oesophagus 0·0825 m.m. long; intestinal bifurcation at about halfway between ventral sucker and anterior end; intestinal coeca in forebody lateral on either side

of ventral sucker and holdfast organ about one-third distance between them and lateral body margins but ventrolateral in hindbody.

Testes post ovarian, tandem, occupying first three-quarters of the length of hindbody. Anterior testis 0·406 m.m. long and 0·527—0·765 m m. broad, asymmetrical, pear-shaped with pointed end near right body margin and broad end pressed against left body margin. Posterior testis 0·357—0·408 m.m. in maximum length, 0·136—0·221 m.m. in length in region of constriction and 0·476—0·799 m.m. broad. symmetrical and hourglass-shaped. Vasa efferentia arise from left anterior corners of testes and unite near the right body margin just in front of pointed end of anterior testis to form the vas deferens. Vesicula seminalis situated just behind posterior testis, large and much coiled. Ovary 0·119—0·153 m.m long and 0·289—0·391 m.m. broad, transversely oval, close in front of anterior testis and just behind body constriction. Oviduct arises from anterodorsal side of ovary. Ootype surrounded by shell gland mass and partly covered by yolk reservoir, intertestecular slightly to right side. Ascending uterus situated to left side extending little in front of hindbody where it bends down towards right side to form the median descending

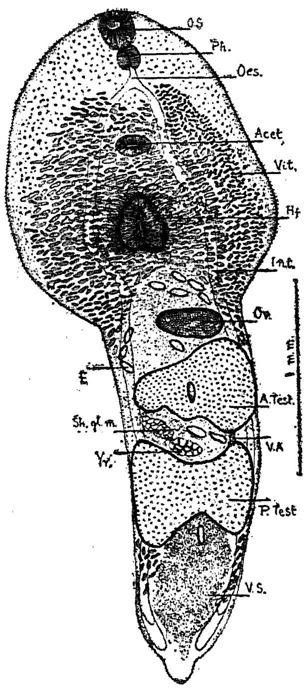

Fig 4.—*Neodiplostomum endynamis* n. sp.

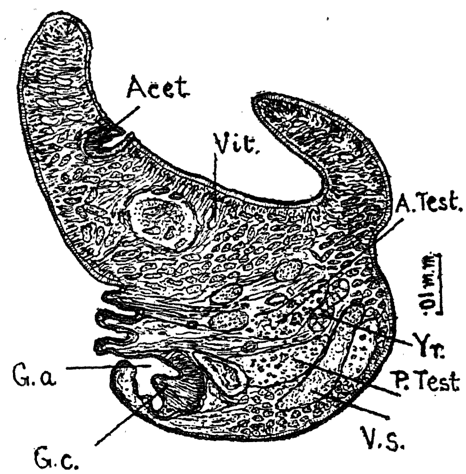

Fig. 5.—*L. S. of Neodiplostomum endynamis* n. sp.

(Lettering as in Fig. 1)

uterus. Ova 0·1075—0·11 m.m. long and 0·037–0·053 m.m. broad, 13 to 15 in number. Genital cone well-developed, highly muscular; eversible filling almost the entire genital atrium when not protruded. Genital opening dorsal little in front of hinder end. Body wall just in front of genital opening provided with three small muscular papillae. Vitellaria composed of numerous follicles, profusely developed in forebody and poorly developed in hindbody, extending from just behind intestinal bifurcation upto a little in front of blind ends of intestinal coeca and restricted to ventrolateral margins in hindbody.

Discussion.—The presence of a well-developed genital cone decides the position of this species under the subgenus *Conodiplostomum* Dubois under which so far no other species of Neodiplostomum from the host birds of the order Cuculi has been recorded. Of all the species of the subgenus *Conodiplostomum* in which both the testes are as a rule symmetrical this species resembles closely *N. perlatum* (Ciurea, 1919) Ciurea 1929, *N. rufeni* n. sp. and *N. muthiari* n. sp. in the asymmetrical condition of its anterior testis, absence of the prepharynx and in the short size of the oesophagus. It differs from *N. perlatum* in the shape of the forebody and hindbody. The forebody is oval and shorter than the hindbody in the new species whereas in *N. perlatum* it is longer or nearly equal to the hindbody with lateral margin much folded. Moreover in *N. perlatum* the body is divided into two parts by a strong constriction unlike that in the new species. The oral sucker in the new species is larger than the pharynx unlike that in *N. perlatum* and the vitellaria in the new species commence, in front of the ventral sucker and are slightly developed in the hindbody confined only to edges, whereas in *N. perlatum* they commence behind the ventral sucker and are strongly developed in the hindbody overlapping the gonads. *N. Eudynamis* n. sp. is distinguished from *N. rufeni* n. sp. by the shape and size of the body, relative width of the forebody and the hindbody, large size of the oral and ventral suckers and the holdfast organ. In *N. eudynamis* n. sp. the vesicula seminalis is much larger and coiled and the vitellaria poorly developed in the hindbody restricted only to lateral margins and not extending towards the median plane like those of *N. rufeni* n. sp. The hosts also belong to different orders of birds, *N. rufeni* n. sp. being parasitic in Accipetres and *N. eudynamis* n. sp. parasitic in Cuculi. *N. muthiari* n. sp. which resembles this species in the poor development of the vitellaria in the hindbody can be easily separated from it on account of the size and shape of the body, relative sizes of the anterior and posterior regions, small size of the holdfast organ, shape and size of the testes, slightly developed condition of the genital cone, smaller number of ova and the host belonging to the order Pici.

Host :—*Eudynamis Scolopaceous.*

Habitat :—Small intestine.

Locality :—Allahabad (U.P., India).

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THE GOLGI BODIES AND THE SECRETION OF FAT DROPLETS IN THE EGGS OF *GALLUS BANKIVA*

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SUMMARY

Free fat spherules, showing no connection with the Golgi elements, are found in the oocytes of Gallus bankiva. A few fat bodies are associated with the Golgi elements, but whether this association is real or fortuitous, due to a crowding together of the Golgi elements and fat bodies in certain parts of the eggs, is not definitely determined, although the latter view seems more probable in consideration of the fact that in younger eggs such associations are rare, if not altogether wanting.

Most vertebrate and invertebrate eggs are found to contain a certain amount of fat bodies in the form of well-marked spherules. About the origin of these fat bodies two mutually exclusive views have been held. Some authors have maintained that fat droplets are formed in the ground cytoplasm, uninfluenced by any of the cytoplasmic components. Others have shown that they are produced by the transformation of, or in direct association with, the Golgi bodies. Rarely the mitochondria (Hosselet, 1930) and nucleolar extrusions (Rao, 1927; Subramaniam and Aiyar, 1936) have been held responsible for the formation of fat. Jägersten (1935) expresses himself strongly on the point. He writes, " Die Entstehung der Fettkügelchen ist von vielen Forschern mit den Golgielementen, den Mitochondrien oder anderen geformeten Elementen in der Zelle verknüpft worden; man hat dabei angenommen, dass sie unmittelbar in Fettkügelchen umgewandelt werden oder auch, dass sie mehr indirekt wirksam sind. Ich habe keine Belege für die Richtigkeit dieser Auffassungen gefunden, sondern im Gegenteil bei meinem Material mit Sicherheit feststellen können, dass keine Umwandlung der genannten Elemente in Fettkügelchen stattfindet. Diese entstehen unmittelbar im Grundcytoplasma ohne nachweisbaren Zusammenhang mit anderen geformten Gebilden." Similarly Payne (1932), who investigated the cytoplasmic components of the eggs of many insects, found no indication that the Golgi elements are in any way involved in the production of fat bodies. He writes, " As far as my observations go, I have found no evidence which would lead me to conclude that the Golgi bodies are concerned either directly or indirectly in either yolk or fat formation." Saguchi, in his account of the cytoplasmic structures of the Amphibian eggs (1932), does not mention any

relationship between the Golgi bodies (or any other inclusions) and fat spherules. On the other hand, he clearly mentions that, "Die Fettkügelchen stellen ein von aussen aufgenommenes Rohmaterial dar, ..." Hibbard and Parat (1928), likewise, do not ascribe the production of fat to the direct or indirect intervention of any cytoplasmic component. They simply mention "gouttes d'huile volumineuses." Harvey too has consistently described the formation of fat bodies independently in the ground cytoplasm.

On the other hand, there is a fairly respectable body of opinion that favours the other view. Gatenby and Woodger (1920) state, "It seems quite probable that the diffuse Golgi elements actively take part in the formation of yolk bodies;......" (in Helix and Limnaea). They mention further that "in the case of Patella the Golgi apparatus provides most of the yolk spheres of the full-grown egg, ..." Ludford (1921) found that in the eggs of Patella the pieces of archoplasm become loaded with fat. Fat bodies also develope under the influence of several Golgi bodies that collect together at many places. Golgi rods are, in such a case, found sticking to the surface of the fat bodies. Brambell (1924) described a transformation of the Golgi elements into fat (his Golgi-yolk or G-yolk) in the eggs of Helix. In Patella Golgi-yolk is formed, according to this author, just as Ludford described it. That Brambell's Golgi-yolk is fatty in nature is evident from the fact that in the centrifuged oocytes of both Helix aspersa and Patella vulgata the Golgi-yolk bodies occupy the pole opposite to the one occupied by the mitochondria. Subsequent research has made it abundantly clear that this pole is occupied by the fat bodies. Additional support to this view is lent by the fact that "The upper layer consisted of spheres which go black or dark grey in Champy's fluid, but only appeared as vacuoles in Da Fano preparations". Brambell himself thought these bodies to be fatty. He wrote, "I think from their reactions that these Golgi-yolk spheres of Helix and Patella are fatty in nature, and that they probably do not contain protein substances." Similarly the yolk in the oocyte of Limnaea which, according to Gatenby and Woodger, is formed by the Golgi elements, is fatty, as on centrifuging it occupies a pole obviously centripetal (Gatenby, 1919). Besides these authors, Bhattacharya and his pupils, Nath and his collaborators, Gresson, Subramaniam and Aiyar and some others have shown a direct participation of the Golgi elements in the production of fat bodies. Kater (1928) records an intimate association between the Golgi elements and fat bodies.

In considering the significance of such an association of the Golgi bodies and the fat spherules, it is necessary to remember Payne's criticism, "Neither does contact between Golgi bodies and yolk and fat-spheres supply convincing evidence that the Golgi bodies contribute to their formation. As the cell becomes crowded with fat and yolk, such contacts are inevitable." Nevertheless an intimate contact

between the Golgi elements and the fat droplets would be strongly suggestive of a participation of the Golgi elements in the origination of the fatty material. Such an association, if real, cannot necessarily be confined to the older eggs where a crowding of the inclusions is most likely to occur. The most important point to determine is whether an unquestionable morphological demonstration of an intimate contact between the Golgi elements and fat bodies has been made. Ludford and Brambell founded their conclusions on the study of osmic-treated material. To obtain a decision on this matter Bhattacharya and his pupils also rely on methods involving the use of osmic acid, and so do Gresson and Kater. In the eggs of the spider, crossopriza, Vishwa Nath (1928) finds "Golgi vacuoles" in eggs prepared according to Mann-Kopsch method and kept in turpentine (Fig. 11). The Golgi elements have "a sharp black rim and a central clear substance." Nath's figures convey the impression that when the fatty portion of the Golgi complex is dissolved out in turpentine there remains a black rim or cortex. Similarly vacuolar bodies with dark rims and light centres are figured by Bhatia and Nath (1931) in their paper on the crustacean oogenesis (Text-Fig. 5). These bodies are seen on examining a portion of the contents of a ripe oocyte kept in 2 per cent. osmic acid for 10 minutes. Bhatia and Nath write, "The smallest Golgi elements (GE) show a dark rim and a very light brownish centre. The bigger ones (GE′) show a dark brownish centre, and the still bigger ones (GE′′) show a still darker centre. The biggest ones (GE′′′) appear uniformly black, as their interior and the rim are blackened to the same extent. From these facts it is to be concluded that the Golgi elements not only grow enormously in size but also become more and more fatty. The fully grown Golgi elements may be called the fatty yolk." Nath and Nangia (1931) observe a similar behaviour of the Golgi bodies in the oocytes of Ophiocephalus. "If a very advanced oocyte measuring 0·97 m.m. is ruptured after only ten minutes' osmication and its contents studied under the microscope, one is driven to the conclusion that the small Golgi vesicles give rise to fatty yolk by a process of growth and deposition of fat inside their interior (fig. 19)." Nath and his collaborators come to the same conclusion on their study of the oogenesis of some other animals. Now Schlottke (1931) thinks that Nath (1929) and Bhandari and Nath (1930), as also Gresson (1929), have described fat as Golgi apparatus. Jägersten (1935) also holds this view. He writes, "Die Fettkügelchen in der Zelle sind Gebilde, von denen kaum anzunehmen gewesen wäre, dass man sie mit Golgielementen verwechseln kann.......Derartiger Irrtümer hat sich besonders Nath und seine Mitarbeiter schuldig gemacht. Ich habe......zwei Fälle (Palaemon und Rana) behandelt, bei welchen derartige Verwechslungen vorgekommen sind. Weitere solche Verwechslungen sind meiner Ansicht nach wenigstens von folgenden Forschern gemacht worden: Nath (1928) bei Crossopriza, Nath (1929) bei Culex, Nath (1930) bei Pheretima, Nath und Husain (1928) bei Otostigmus, Nath und Mehta

(1929) bei Luciola, Nath und Mohan (1929) bei Periplaneta, Nath und Nangia (1931) bei Knochenfischen, Bhandari und Nath (1930) bei Dysdercus, Sharga (1928) bei Pheretima, Gresson (1929 bund 1931) bei Tenthrediniden und Periplaneta, Rai (1930) bei Ostrea." Further he advances the argument that 'Es besteht somit kein Zweifel darüber, dass die in frischem, lebendem und die in kurze Zeit mit Osmiumsäure behandeltem Material beobachteten und von den genannten Forschern als Golgielemente aufgefassten Gebilde nichts anderes als gewöhnliche Fettkügelchen sind." Jägersten admits that these authors have also possibly observed the real Golgi bodies in silver and some osmic preparations. What he emphasizes is the fact that osmicated fat bodies have been described as Golgi elements.

That Nath and his collaborators have been dealing, in these cases, with fat bodies cannot be denied. The point which Nath emphasizes is that fat accumulates inside the Golgi vesicles. That is to say, a fairly big fat spherule is not a homogeneous structure, but, on the contrary, it consists of two distinct elements, the Golgi body in the form of an osmiophil rim and free fat inside it. The present writer (1934) found that on subjecting osmicated eggs of Passer domesticus to the action of turpentine the fat bodies were dissolved out within a few minutes, but the empty vacuoles, in nearly each case, carried some osmiophil granules or crescents, which were considered to be the Golgi bodies (Fig. 13). I have observed the same phenomenon in the eggs of Anthia sexguttata. Srivastava and Bhattacharya (1935) and Das (1931) concluded the participation of the Golgi elements in the production of fat bodies on more or less similar grounds.

All these authors have drawn their conclusions from a study of osmicated material. Now it is precisely in this connection, *i.e.*, the persistence of a black rim or cortex after the fat has been extracted from osmicated eggs with turpentine or other fat-solvents, that the following remarks of Hoerr (1936) are of the greatest significance. Hoerr writes:—"A great many unjustified deductions have been drawn from observations on material in which there has been incomplete oxidation of lipin droplets with the resulting blackened circles and crescents around paler centres or vacuoles. It should be obvious that these appearances are produced by an incomplete oxidation of the lipin droplet as the osmic acid penetrates from without. If the centre of the lipin droplet has not been sufficiently oxidized so as to be rendered completely insoluble in the solvents employed, then the resulting picture naturally is a blackened circle with an empty centre (fig. 2). That such a result does not mean a core of a more soluble lipin surrounded by a circumference of more insoluble lipins or neutral fat (as Cramer and Gatenby, '28, have concluded) can be proved by taking comparable sections and fixing them in osmic acid for varying lengths of time. If longer periods of fixation are employed, the droplets that appeared in earlier fixation as circles will be seen to be completely blackened. Here, as in any chemical process, the rate of reaction and

the speed of penetration are highly important." In face of this criticism it becomes necessary, in order to trace the connection between the Golgi elements and fat bodies, to use some other method than Ludford followed by the extraction of fat in turpentine.

The present work was undertaken in order to observe the results obtained by the use of such a method. It may be mentioned here that Aoyama followed by Sudan IV, as recommended by Gatenby (The microtomist's vade-mecurn, by Gatenby, J.B., and Painter, T.S., London, 1937) for a satisfactory solution of this problem, could not be employed for want of a freezing microtome. The procedure adopted was as follows :—

1. Fixation of the ovary (of Gallus bankiva) in 1·5 per cent. osmic acid for 24 hours.

2. Washing in two changes of distilled water for 5—10 minutes.

3. Keeping the material in 2 per cent. silver nitrate solution in dark for 48 hours.

4. Washing in distilled water for 5 – 10 minutes.

5. Keeping the material in the reducing solution (hydroquinone, 1·5 gram ; neutral formalin, 15 cc. ; distilled water, 100 cc.; sodium sulphite ·5 gram) for 24 hours.

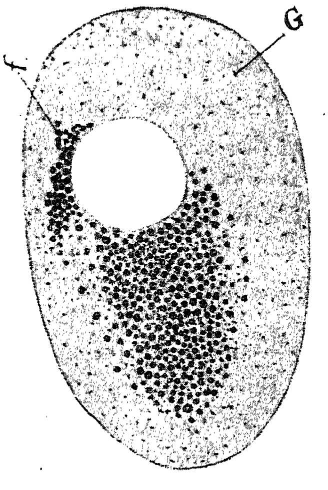

Fig. 1.—A young oocyte, showing fat bodies and Golgi elements

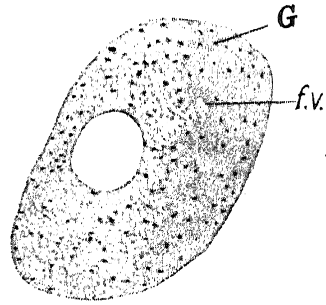

Fig. 2 —A slightly younger oocyte, showing fatty vacuoles and Golgi elements.

G., Golgi body ; *f.*, fat body ; *f. v.*, fatty vacuole.

6. Dehydration in ascending grades of alcohols, clearing in xylol, and imbedding in paraffin wax.

Sections were cut 5μ thick. Some were mounted straight in balsam and examined and sketched immediately. It is unnecessary to extract the fat with turpentine, as the xylol in balsam removes it completely within twenty-four hours. Some sections were also brought to water and treated with 1 per cent. potassium permanganate till the blackened fat spherules were completely decolorized. The slides were then flooded with oxalic acid and subsequently kept in running water for ten minutes, after which they were stained with Sudan IV.

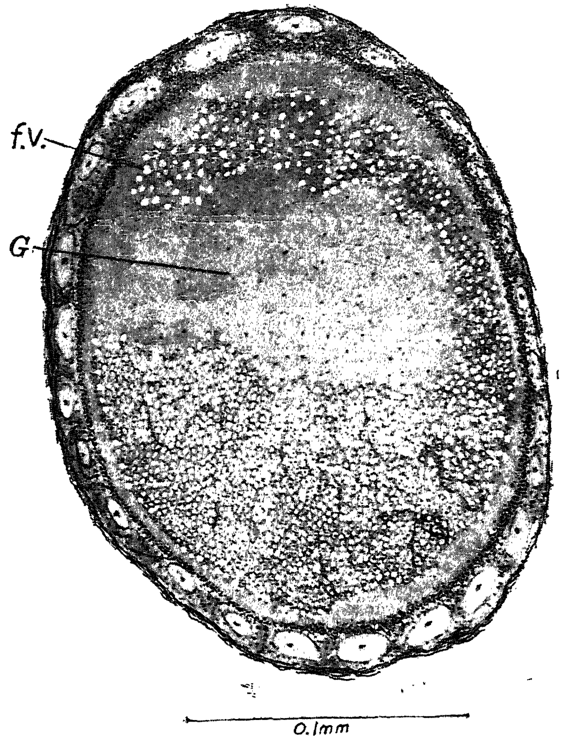

Fig. 3
(Lettering as in Fig. 1.)
Fig. 3.—A bigger oocyte, showing fatty vacuoles and Golgi elements.

Figure 1 represents an oocyte sketched immediately after being mounted. The egg contains a good many fatty spherules gone black in osmic acid. In addition to these there are numerous argentophil Golgi elements, crescentic or spherical, distributed irregularly over the entire stretch of the cytoplasm. Figure 2 shows a similar condition except that the place of fat bodies is taken by empty vacuoles due to the dissolution of the former. Figure 3 represents an egg sketched twenty-four hours after it was mounted. The xylol in balsam has dissolved out the

fatty spherules, which consequently appear as empty vacuoles. The argentophil Golgi elements are irregularly scattered in the cytoplasm. They appear more numerous in that part of the egg which contains the greatest number of fatty vacuoles and, therefore, wears a frothy look. At places they are found between and inside the vacuoles. There are, however, a good many fatty vacuoles that exhibit absolutely no connection with the Golgi elements. In Fig. 3 most of the upper vacuoles are of this nature. Fig. 4 represents a portion of an oocyte drawn at a higher magnification. It shows that most of the fatty vacuoles in this particular region are not associated with the Golgi elements. Another fact of importance is that in no case do the fatty vacuoles show a complete argentophil cortex, which proves that the fat bodies are not vesicular in form, composed of an outer Golgi substance (the rim) with free fat in the interior.

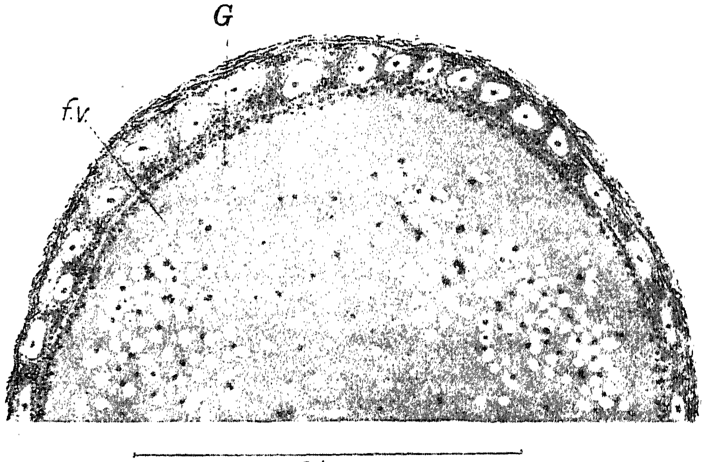

Fig. 4

(Lettering as in Fig. 1)

Fig. 4.—Part of an oocyte, showing fatty vacuoles and Golgi elements.

It is easy to see that in the eggs of Gallus there do exist fatty spherules which are produced in the ground cytoplasm independently of the Golgi elements. Most of the fat bodies are assuredly of this nature. There are, however, some fat vacuoles which are, in some way or other, associated with the Golgi bodies. Whether this association is real and denotes a functional significance of the Golgi elements, or fortuitous, caused by the crowding together of the fat bodies and the Golgi elements, is a little difficult to decide. It is probably significant that in younger eggs (Fig. 2) carrying fewer Golgi elements and fewer fat bodies such associations hardly exist. The important fact is that free fat spherules bearing no connection with the Golgi bodies normally occur in the eggs of Gallus. It is possible that all fat bodies are of this nature, though this is by no means satisfactorily established.

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ON THE NEW CONGRUENCE PROPERTIES OF THE ARITHMETIC FUNCTION T(n)

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Ramanujan[1] assumed some of the Congruence properties of the arithmetic function T(n) which were proved by Darling[2] and Mordell.[3] I have proved[4] the Congruence properties of the arithmetic function T(n) regarding the modulii 3, 4, 9, 12, 11, 25, 691. Here I shall prove the Congruence properties for modulii 8^0, 8^n, 7^n, 56, 56^n, 40. We know that $(1-x)^2=1-x^2+2J$ where J is an integral power series in x whose coefficients are integers.

Therefore $(1-x)^8=(1-x^2)^4+8\text{ J}$. Hence[5] $\sum_1^{\infty} T(n)\,x^{n-1}=\{(1-x)$

$$(1-x^2)\,(1-x^3)\ldots\}^{24}=\{(1-x^2)\,(1-x^4)\ldots\}^{12}+8\text{ J.}=[1-3x^2+5x^6\ldots]^4+8\text{ J.}$$

Since the terms containing odd powers of x are absent in $[1-3x^2+5x^6\ldots]^4$ we have $T(2n)\equiv 0 \pmod 8$.

Since [6] $T(5n)\equiv 0 \pmod 5$ and $T(7n)\equiv 0 \pmod 7$ we can prove easily $T(14n)\equiv 0 \pmod{56}$ and $T(10n)\equiv 0 \pmod{40}$. Again[7] $T(p^{n+1}m)=T(p)\ T(p^n m)-p^{11}\ T(p^{n-1}m)$ where p is a prime number and $(p_1 m)=1$. Hence $T(2^{n+1}m)=T(2)x$ $T(2^n m)-2^{11}\ T(2^{n-1}m)$. Therefore $T(2^2m)=T(2)\ T(2m)-2^{11}\ T(m)$. Hence

$T(2^2m) \equiv 0 \pmod{8^2}$. Similarly $T(2^3m) \equiv 0 \pmod{8^3}$. Hence $T(2^n m) \equiv 0 \pmod{8^n}$. Similarly $T(7^n m) \equiv 0 \pmod{7^n}$ and $T(14^n m) \equiv 0 \pmod{56^n}$. The Congruence property for 5^n has been included in a paper for congruence properties of some arithmetic functions which will be published shortly.

References

1. Collected papers of Ramanujan by G. H. Hardy, p. 230.

2. Darling, " Proofs of certain identities and congruence ennunciated by Ramanujan" Proc. London Math. Soc. (2), Vol. 19 (1921).

3. Mordell, " Note on certain modular relations considered by Messrs Ramanujan, Darling and Rogers," Proc. London Math. Soc. (2), Vol. 20 (1922).

4. To be published in the Proc. London Math. Soc.

5. Ramanujan, *l. c.*, p., 151.

6. Ramanujan, *l. c.*, p. 230.

7. Mordell, "On Mr. Ramanujan's empirical expansions of Modular functions." Proc. Camb. Phil. Soc., Vol. 19 (1919), p. 119.

Or we can prove this from the formulæ $p^{\frac{-11}{2}a}\, T(p^a) = \frac{\sin(1+a)\,\theta_p}{\sin \theta_p}$ where $2 \cos\theta_p = p^{\frac{-11}{2}} T(p)$.

Cf. p. 153 of the collected papers of Ramanujan.

MOTION OF AN INCOMPRESSIBLE FLUID WITH VARYING CO-EFFICIENT OF VISCOSITY, GIVEN BY $\mu=\mu_0+\varepsilon_1 x$, FOR POSITIVE VALUES OF x. PART II

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In a previous paper, *viz.*, "Motion of an incompressible fluid with varying coefficient of viscosity, given by $\mu=\mu_0+\varepsilon_1 x$, for positive values of x. Part I," we considered the motion after neglecting the inertia terms. Here the inertia terms have been retained up to the first order of small quantities, when u, v and w are small. Here also the law of variation in the co-efficient of viscosity is the same as in the previous paper, *viz.*, $\mu=\mu_0+\varepsilon_1 x$, for positive values of x. At the origin the value of μ is μ_0 and we shall further assume that μ *is constant and equal to zero for negative values of* x and the density of the liquid has been taken constant throughout the motion.

Here also ε_1 has been taken so small that its square and higher powers are neglected. Motion of the fluid at a finite distance and at a great distance from the origin has been considered.

Proceeding as in Paper (I), we get the fundamental equations of motion, when μ is a function of x only, to be

$$\rho\frac{Du}{Dt}=\left\{\rho X-\frac{\partial p}{\partial x}+\tfrac{1}{3}\mu\frac{\partial\theta}{\partial x}+\mu\nabla^2 u\right\}+\frac{\partial\mu}{\partial x}\left(2\frac{\partial u}{\partial x}-\tfrac{2}{3}\theta\right)$$

$$\rho\frac{Dv}{Dt}=\left\{\rho Y-\frac{\partial p}{\partial y}+\tfrac{1}{3}\mu\frac{\partial\theta}{\partial y}+\mu\nabla^2 v\right\}+\frac{\partial\mu}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$\rho\frac{Dw}{Dt}=\left\{\rho Z-\frac{\partial p}{\partial z}+\tfrac{1}{3}\mu\frac{\partial\theta}{\partial z}+\mu\nabla^2 w\right\}+\frac{\partial\mu}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

where X, Y and Z are the extraneous forces in the direction of the axes and

$$\theta\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}.$$

Now in our case, $\mu=\mu_0+\varepsilon_1 x$, where μ_0 and ε_1 are constants.

The equations of motion, in the absence of extraneous forces and on the substitution of the value of μ and $\frac{\partial \mu}{\partial x}$, become in the case of an incompressible fluid

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + (\mu_0+\varepsilon_1 x)\ \nabla^2 u + \varepsilon_1 \left(2\frac{\partial u}{\partial x}\right)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + (\mu_0+\varepsilon_1 x)\ \nabla^2 v + \varepsilon_1 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + (\mu_0+\varepsilon_1 x)\ \nabla^2 w + \varepsilon_1 \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

with $\theta=0$.

Now we consider two cases when u, v and w are small, the first one being at a finite distance and the other at a great distance from the origin. We shall further assume that ε_1 is small and if we retain terms up to the first order of small quantities, we may neglect $\varepsilon_1\left(2\frac{\partial u}{\partial x}\right)$, $\varepsilon_1\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ & $\varepsilon_1\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$ from the equations of motion. Then we have to investigate whether we can neglect $\varepsilon_1 x\nabla^2 u$, $\varepsilon_1 x\nabla^2 v$ and $\varepsilon_1 x\nabla^2 w$ from the said equations of motion when x is sufficiently large. Again we may neglect all terms in $\rho\frac{Du}{Dt}$, $\rho\frac{Dv}{Dt}$ and $\rho\frac{Dw}{Dt}$ except $\rho\frac{\partial u}{\partial t}$, $\rho\frac{\partial v}{\partial t}$ and $\rho\frac{\partial w}{\partial t}$.

When we are taking into account the motion only at a finite distance from the origin, $\varepsilon_1 x\nabla^2 u$, etc., become small quantities of the second order and we may therefore neglect them. The equations of motion then become

$$\left.\begin{aligned} \rho\frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + \mu_0\nabla^2 u \\ \rho\frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial y} + \mu_0\nabla^2 v \\ \rho\frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} + \mu_0\nabla^2 w \end{aligned}\right\}$$

with
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This case has been solved.[1]

Now for the motion at a great distance from the origin μ changes appreciably and we cannot neglect $\varepsilon_1 x \nabla^2 u$, etc., from the equations of motion which become

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu_0+\varepsilon_1 x}{\rho}\nabla^2 u\\ \frac{\partial v}{\partial t}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu_0+\varepsilon_1 x}{\rho}\nabla^2 v\\ \frac{\partial w}{\partial t}&=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu_0+\varepsilon_1 x}{\rho}\nabla^2 w\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we assume that u, v, w all vary as $e^{\alpha t}$, the equations (1) and (2) can be written as

$$\left.\begin{aligned}u&=-\frac{1}{\rho_1}\frac{\partial p}{\partial x_1}+\frac{\varepsilon_1}{\rho_1}x_1\nabla^2 u\\ v&=-\frac{1}{\rho_1}\frac{\partial p}{\partial y}+\frac{\varepsilon_1}{\rho_1}x_1\nabla^2 v\\ w&=-\frac{1}{\rho_1}\frac{\partial p}{\partial z}+\frac{\varepsilon_1}{\rho_1}x_1\nabla^2 w\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}1)$$

with

$$\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}1)$$

where $\rho_1=\alpha\rho$, $\mu_0+\varepsilon_1 x=\varepsilon_1' x_1$ and ∇^2 now means $\dfrac{\partial^2}{\partial x_1^{\,2}}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$.

In a physical problem, due to dissipative forces, if the velocity continually diminishes, α must be negative.

We write (1·1) in the following form :—

$$\left.\begin{aligned}(\varepsilon_1 x_1\nabla^2-\rho_1)\,u&=\frac{\partial p}{\partial x_1}\\ (\varepsilon_1 x_1\nabla^2-\rho_1)\,v&=\frac{\partial p}{\partial y}\\ (\varepsilon_1 x_1\nabla^2-\rho_1)\,w&=\frac{\partial p}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}2)$$

with

$$\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}1)$$

or

$$\left.\begin{aligned}\left(\nabla^2-\frac{\rho_1}{\varepsilon_1 x_1}\right)u&=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial x_1}\\ \left(\nabla^2-\frac{\rho_1}{\varepsilon_1 x_1}\right)v&=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial y}\\ \left(\nabla^2-\frac{\rho_1}{\varepsilon_1 x_1}\right)w&=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1\cdot 3)$$

Differentiating the first equation with respect to x_1, the second with respect to y, and the third with respect to z in (1·3) and adding and then using (2·1) we get

$$\frac{\rho_1}{\varepsilon_1 x_1{}^2}u=\frac{1}{\varepsilon_1 x_1}\nabla^2 p-\frac{1}{\varepsilon_1 x_1{}^2}\frac{\partial p}{\partial x_1}$$

or

$$\rho_1 u=x_1\nabla^2 p-\frac{\partial p}{\partial x_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this value of $\rho_1 u$ in the first equation of (1·2) we get the equation for determining 'p' to be

$$(\varepsilon_1 x_1\nabla^2-\rho_1)\left(x_1\nabla^2 p-\frac{\partial p}{\partial x_1}\right)=\rho_1\frac{\partial p}{\partial x_1} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$\varepsilon_1 x_1\nabla^2\left(x_1\nabla^2 p-\frac{\partial p}{\partial x_1}\right)-\rho_1 x_1\nabla^2 p+\rho_1\frac{\partial p}{\partial x_1}=\rho_1\frac{\partial p}{\partial x_1}$$

or

$$\nabla^2\left(x_1\nabla^2 p-\frac{\partial p}{\partial x_1}\right)-\frac{\rho_1}{\varepsilon_1}\nabla^2 p=0 \quad . \quad . \quad . \quad . \quad (4\cdot 1)$$

Now

$$\frac{\partial}{\partial x_1}(x_1\nabla^2 p)=x_1\frac{\partial}{\partial x_1}\nabla^2 p+\nabla^2 p$$

$$\frac{\partial^2}{\partial x_1{}^2}(x_1\nabla^2 p)=\frac{\partial}{\partial x_1}\left(x_1\frac{\partial}{\partial x_1}\nabla^2 p+\nabla^2 p\right)=x_1\frac{\partial^2}{\partial x_1{}^2}\nabla^2 p+2\frac{\partial}{\partial x_1}\Delta^2 p$$

Hence simplifying (4·1) we get

$$x_1\nabla^2(\nabla^2 p)+2\frac{\partial}{\partial x_1}\nabla^2 p-\frac{\partial}{\partial x_1}\nabla^2 p-\rho_0\Delta^2 p=0,\text{ where }\frac{\rho_1}{\varepsilon_1}=\rho_0.$$

or

$$x_1\nabla^2(\nabla^2 p)+\frac{\partial}{\partial x_1}\nabla^2 p-\rho_0\nabla^2 p=0 \quad . \quad . \quad . \quad . \quad (4\cdot 2)$$

Put $\nabla^2 p=\mathrm{P}$ . . . . (5)

so that

$$x_1\nabla^2\mathrm{P}+\frac{\partial \mathrm{P}}{\partial x_1}-\rho_0\mathrm{P}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let $P=P_1 P_2$, where P_1 is a function of x_1 alone and P_2 that of y and z only. Substituting in (6) we get

$$\left(P_2\frac{d^2P_1}{dx_1{}^2}+P_1\nabla_1{}^2P_2\right)+\frac{1}{x_1}P_2\frac{dP_1}{dx_1}-\rho_0P_1P_2=0, \text{ where } \nabla_1{}^2\equiv\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

or
$$\frac{1}{P_1}\frac{d^2\rho_1}{dx_1{}^2}+\frac{1}{P_2}\nabla_1{}^2P_2+\frac{1}{P_1}\frac{1}{x_1}\frac{dP_1}{dx_1}-\frac{\rho_0}{x_1}=0$$

This can be satisfied if we write

$$\frac{1}{P_1}\frac{d^2P_1}{dx_1{}^2}+\frac{1}{P_1}\cdot\frac{1}{x_1}\frac{dP_1}{dx_1}-\frac{\rho_0}{x_1}-k=0 \text{ and } \frac{1}{P_2}\nabla_1{}^2P_2+k=0,$$

k being an arbitrary constant.

or
$$x_1\frac{d^2P_1}{dx_1{}^2}+\frac{dP_1}{dx_1}-(\rho_0+kx_1)\,P_1=0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and
$$\nabla_1{}^2P_2+kP_2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

First we consider (7) :

$$x_1\frac{d^2P_1}{dx_1{}^2}+\frac{dP_1}{dx_1}-\left(\rho_0+kx_1\right)P_1=0$$

Let $P_1=A_1 x_1{}^{m_1}+A_2 x_1{}^{m_2}+A_3 x_1{}^{m_3}+ \quad . \quad . \quad . \quad +A_n x_1{}^{m_n}+ \quad . \quad . \quad .$

Substituting we get

$$\left.\begin{array}{l}
A_1 m_1{}^2x_1{}^{m_1-1}-A_1\rho_0x_1{}^{m_1}-A_1 k x_1{}^{m_1+1}\\
+A_2 m_2{}^2x_1{}^{m_2-1}-A_2\rho_0x_1{}^{m_2}-A_2 k x_1{}^{m_2+1}\\
+A_3 m_3{}^2x_1{}^{m_3-1}-A_3\rho_0x_1{}^{m_3}-A_3 k x_1{}^{m_3+1}\\
+\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
+A_{n-2} m^2{}_{n-2} x_1{}^{m_{n-2}-1}-A_{n-2}\rho_0 x_1{}^{m_{n-2}}-A_{n-2} k x_1{}^{m_{n-2}+1}\\
+A_{n-1} m^2{}_{n-1} x_1{}^{m_{n-1}-1}-A_{n-1}\rho_0 x_1{}^{m_{n-1}}-A_{n-1} k x_1{}^{m_{n-1}+1}\\
+A_n m^2{}_n x_1{}^{m_n}-A_n\rho_0 x_1{}^{m_n}-A_n k x_1{}^{m_n+1}\\
+\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{array}\right\}\equiv 0$$

This gives $m_1{}^2=0$, *i.e.*, $m_1=0$, $\quad m_1=m_2-1 \quad \therefore \quad m_2=m_1+1$

$m_2=m_3-1 \quad \therefore \quad m_3=m_2+1=m_1+2$

$m_n=m_1+(n-1).$

$$-A_1\rho_0+A_2 m_2{}^2=0$$
$$-A_{n-2} k-A_{n-1}\rho_0+A_n m_n{}^2=0, \qquad \text{for } n\geq 3.$$

or
$$\frac{A_n}{A_{n-2}}=\frac{A_{n-1}}{A_{n-2}}\cdot\frac{\rho_0}{m_n^2}+\frac{k}{m_n^2}$$

or
$$\frac{A_n}{A_{n-2}}=\frac{A_{n-1}}{A_{n-2}}\frac{\rho_0}{(m_1+n-1)^2}+\frac{k}{(m_1+n-1)^2}$$

Taking limits as $n\to\infty$, we find

$$\frac{A_n x_1}{A_{n-1}}\to 0.$$

Hence the series is absolutely and uniformly convergent for all values of x_1.

Now
$$A_2=\frac{A_1}{m_2^2}\rho_0=\frac{A_1}{(m_1+1)^2}\rho_0$$

$$A_3 m_3^2=A_2\rho_0+A_1 k=A_1\left\{k+\frac{\rho_0^2}{(m_1+1)^2}\right\}$$

$$\therefore\ A_3=\frac{1}{(m_1+2)^2}\left\{k+\frac{\rho_0^2}{(m_1+1)^2}\right\}A_1,$$

$$A_4(m_1+3)^2=A_3\rho_0+A_2 k=A_1\left[\frac{k\rho_0}{(m_1+1)^2}+\frac{\rho_0}{(m_1+2)^2}\left\{k+\frac{\rho_0^2}{(m_1+1)^2}\right\}\right]$$

$$\therefore\quad A_4=\frac{1}{(m_1+3)^2}\left\{\frac{k\rho_0}{(m_1+1)^2}+\frac{\rho_0}{(m_1+2)^2}\cdot\overbrace{k+\frac{\rho_0^2}{(m_1+1)^2}}\right\},\ \text{etc.}$$

$$\therefore\ P_1=A_1x_1^{m_1}\left[1+\frac{\rho_0}{(m_1+1)^2}x_1+\frac{1}{(m_1+2)^2}\left\{k+\frac{\rho_0^2}{(m_1+1)^2}\right\}x_1^2\right.$$
$$\left.+\frac{1}{(m_1+3)^2}\left\{\frac{k\rho_0}{(m_1+1)^2}+\frac{\rho_0}{(m_1+2)^2}\cdot\overbrace{k+\frac{\rho_0^2}{(m_1+1)^2}}\right\}x_1^3+\ \ldots\right]$$

where A_1 is an arbitrary constant.

Since the two roots of m_1 are zero, the solution of (7) is given by

$$P_1=B'_1\,P'_1+B'_2\,P''_1$$

where
$$P'_1=\Big[P_1\Big]_{m_1=0}\ \text{and}\ P''_1=\left[\frac{\partial P_1}{\partial m_1}\right]_{m_1=0}.$$

$$P'_1=A_1\left[1+\rho_0x_1+\frac{1}{2^2}(k+\rho_0^2)\,x_1^2+\frac{1}{3^2}\left\{k\rho_0+\frac{\rho_0}{2^2}(k+\rho_0^2)\right\}x_1^3+\ \ldots\right]$$

For P_2 we consider (8).

$$\nabla_1^2 P_2+kP_2=0$$

whereby[2] $$P_2 = B'_s J_s\left(\beta_1\omega\right)\begin{matrix}\cos\\ \sin\end{matrix}\Big\} s\phi, \qquad \text{if } k=\beta_1^2$$

or $$B''_m I_m\left(\beta_2\omega\right)\begin{matrix}\cos\\ \sin\end{matrix}\Big\} m\phi, \qquad \text{if } k=-\beta_2^2$$

Now we consider the equation

$$\nabla^2 p = P = P_1 P_2.$$

Put $p = P_2 p_1$, where p_1 is a function of x_1 only so that

$$P_2 \frac{d^2 p_1}{dx_1^2} + p_1 \nabla_1^2 P_2 = P_1 P_2$$

or $$P_2\left(\frac{d^2 p_1}{dx_1^2} - k p_1\right) = P_1 P_2, \qquad \text{by (8).}$$

or $$\frac{d^2 p_1}{dx_1^2} - k p_1 = P_1$$

$$\therefore \quad p_1 = c_1 e^{\sqrt{k}\,x_1} + c_2 e^{-\sqrt{k}\,x_1} + \frac{1}{D^2-k} P_1$$

Now $\frac{1}{D^2-k} P_1$ can be easily found[3].

$\therefore \quad p = \Sigma p_n + p_1 P_2$, where p_n is a solid harmonic of degree n.

Turning to (1·3) and (2·1) we have

$$\left(\nabla^2 - \frac{\rho_1}{\varepsilon_1 x_1}\right) u = \frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial x_1}$$

$$\left(\nabla^2 - \frac{\rho_1}{\varepsilon_1 x_1}\right) v = \frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial y}$$

$$\left(\nabla^2 - \frac{\rho_1}{\varepsilon_1 x_1}\right) w = \frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial z}$$

with $$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

We find

$$\left.\begin{aligned} \rho_1 u &= x_1 \nabla^2 p - \frac{\partial p}{\partial x_1} \quad \ldots\ldots \text{ from (3)} \\ \rho_1 v &= \frac{1}{k} x_1 \frac{dP_1}{dx_1}\cdot\frac{\partial P_2}{\partial y} - \frac{\partial p}{\partial y} \\ \rho_1 w &= \frac{1}{k} x_1 \frac{dP_1}{dx_1}\cdot\frac{\partial P_2}{\partial z} - \frac{\partial p}{\partial z} \end{aligned}\right\} \quad \ldots \quad (A)$$

satisfy the equations of motion.

Since
$$\frac{\partial}{\partial x_1}\left(x_1\nabla^2 p-\frac{\partial p}{\partial x_1}\right)+\frac{\partial}{\partial y}\left(\frac{1}{k}x_1\frac{dP_1}{dx_1}\frac{\partial P_2}{\partial y}-\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{k}x_1\frac{dP_1}{dx_1}\frac{\partial P_2}{\partial z}-\frac{\partial p}{\partial z}\right)$$

$$=x_1P_2\frac{d\rho_1}{dx_1}+\nabla^2 p-\frac{\partial^2 p}{\partial x_1^2}+\frac{x_1}{k}\frac{dP_1}{dx_1}\frac{\partial^2 P_2}{\partial y^2}-\frac{\partial^2 p}{\partial y^2}+\frac{x_1}{k}\frac{dP_1}{dx_1}\frac{\partial^2 P_2}{dz^2}-\frac{\partial^2 p}{\partial z^2}$$

$$=x_1P_2\frac{dP_1}{dx_1}+\frac{x_1}{k}\frac{dP_1}{dx_1}\left(\frac{\partial^2 P_2}{\partial y^2}+\frac{\partial^2 P_2}{\partial z^2}\right)=0 \qquad \text{by (8)}$$

and
$$\nabla^2 v=\frac{1}{\rho_1}\left[\frac{d^2}{dx_1^2}\left(x_1\frac{dP_1}{dx_1}\right)\cdot\frac{1}{k}\cdot\frac{\partial P_2}{\partial y}+\frac{1}{k}x_1\frac{dP_1}{dx_1}\nabla_1^2\frac{\partial P_2}{\partial y}-\frac{\partial}{\partial y}\cdot\nabla^2 p\right]$$

$$-\frac{\rho_1}{\varepsilon_1 x_1}v_1=-\frac{1}{\varepsilon_1 x_1}\left(\frac{1}{k}x_1\frac{dP_1}{dx_1}\frac{\partial P_2}{\partial y}-\frac{\partial p}{\partial y}\right)$$

$$\therefore\ \left(\nabla^2-\frac{\rho_1}{\varepsilon_1 x_1}\right)v_1=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial y}+\frac{1}{\rho_1}\left[\frac{1}{k}\frac{\partial P_2}{\partial y}\left(x_1\frac{d}{dx_1}\frac{d^2P_1}{dx_1^2}+2\frac{d^2P_1}{dx_1^2}\right)+\frac{1}{k}x_1\frac{dP_1}{dx_1}\left(-k\frac{\partial P_2}{\partial y}\right)-P_1\frac{\partial\rho_2}{dy}\right]-\frac{1}{\varepsilon_1 k}\frac{dP_1}{dx_1}\frac{\partial P_2}{\partial y}$$

$$=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial y}+\frac{1}{k\rho_1}\frac{\partial P_2}{\partial y}\left[\frac{d}{dx_1}\left(x_1\frac{d^2P_1}{dx_1^2}+\frac{dP_1}{dx_1}\right)-kx_1\frac{dP_1}{dx_1}-kP_1-\rho_0\frac{dP_1}{dx}\right]$$

$$=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial y}+\frac{1}{k\rho_1}\frac{\partial P_2}{\partial y}\cdot\frac{d}{dx_1}\left[x_1\frac{d^2P_1}{dx_1^2}+\frac{dP_1}{dx_1}-(\rho_0+kx_1)P_1\right]$$

$$=\frac{1}{\varepsilon_1 x_1}\frac{\partial p}{\partial y}, \qquad \text{by (7)}$$

Similar is the treatment with w.

Now we have to find the solution of

$$\left.\begin{aligned}(x_1\nabla^2-\rho_0)\ u^1&=0\\(x_1\nabla^2-\rho_0)\ v^1&=0\\(x_1\nabla^2-\rho_0)\ w^1&=0\end{aligned}\right\} \qquad \ldots\ldots\ldots (9)$$

with
$$\frac{\partial u^1}{\partial x_1}+\frac{\partial v^1}{\partial y}+\frac{\partial w^1}{\partial z}=0 \qquad \ldots\ldots\ldots (10)$$

Using (10) we get $\nabla^2 u^1=0$

whereby from the first equation of (9) we get $u^1=0$ . . . . . (9·A)

so that our equations reduce to

$$\left.\begin{aligned}(x_1\nabla^2-\rho_0)\ v^1=0\\(x_1\nabla^2-\rho_0)\ w^1=0\end{aligned}\right\} \quad \ldots \ldots \quad (9\cdot1)$$

with
$$\frac{\partial v^1}{\partial y}+\frac{\partial w^1}{\partial z}=0 \quad \ldots \ldots \quad (10\cdot1)$$

A solution of (9·1) and (10·1) is given by

$$\left.\begin{aligned}v^1&=-R\frac{\partial S}{\partial z}\\w^1&=-R\frac{\partial S}{\partial y}\end{aligned}\right\} \quad \ldots \ldots \quad (11)$$

where
$$\nabla_1{}^2S+k_1S=0 \quad \ldots \ldots \quad (12)$$

and
$$x_1\frac{d^2R}{dx_1{}^2}-(\rho_0+k_1x_1)R=0 \quad \ldots \quad (13),$$
k_1 being an arbitrary constant.

Let
$$D\equiv x_1\frac{d^2}{dx_1{}^2}-(\rho_0+k_1x_1)$$

We construct an expression $V=c_0x_1{}^{\alpha}+c_1x_1{}^{\alpha+1}+c_2x_1{}^{\alpha+2}+\ldots\ldots$

and find
$$x_1DV=c_0\alpha(\alpha-1)x_1{}^{\alpha}$$

provided
$$c_1\alpha(\alpha+1)=c_0\rho_0 \quad \ldots \ldots \quad (14)$$

and $c_n\,(\alpha+n)\,(\alpha+n-1)=c_{n-1}\,\rho_0+c_{n-2}\,k_1$, for $n\geq2$ $\quad \ldots \ldots \quad (15)$

or
$$\frac{c_n}{c_{n-2}}=\frac{c_{n-1}}{c_{n-2}}\cdot\frac{\rho_0}{(\alpha+n)\,(\alpha+n-1)}+\frac{k_1}{(\alpha+n)\,(\alpha+n-1)}$$

Taking limits as $n\to\infty$, we get $\dfrac{c_n\,x_1}{c_{n-1}}\to0.$

Hence the series is absolutely and uniformly convergent for all values of x_1.

Now our indical equation is $\alpha\,(\alpha-1)=0$, *i.e.*, $\underline{\alpha=1 \text{ or zero}}$.

Therefore, by the method of Frobenius, the two solutions will be

$$R_1=\Big[\ V\ \Big]_{\alpha=1} \quad \text{and} \quad R_2=\left[\frac{\partial V}{\partial\alpha}\right]_{\alpha=0} \quad \ldots \ldots \quad (16)$$

$$C_1=\frac{C_0\rho_0}{\alpha(\alpha+1)}$$

$$C_n=\frac{1}{\alpha+n}\,\frac{1}{\alpha+n-1}\,(C_{n-1}\,\rho_0+C_{n-2}\,k_1),\ \text{for } n\geq2.$$

$$C_2=\frac{1}{(\alpha+2)(\alpha+1)}(C_1\rho_0+C_0k_1)=\frac{C_0}{(\alpha+2)(\alpha+1)}\left\{\frac{\rho_0{}^2}{(\alpha+1)\alpha}+k_1\right\}.$$

$$C_3=\frac{1}{(\alpha+3)(\alpha+2)}(C_2\rho_0+C_1k_1)=\frac{C_0}{(\alpha+3)(\alpha+2)}\left\{\frac{\rho_0}{(\alpha+2)(\alpha+1)}\overbrace{\frac{\rho_0{}^2}{(\alpha+1)\alpha}+k_1}+\frac{\rho_0\ k_1}{(\alpha+1)\alpha}\right\}, \text{ etc.}$$

$$\therefore\ V=C_0x_1{}^{\alpha}\left[1+\frac{\rho_0}{(\alpha+1)\alpha}x_1+\frac{1}{(\alpha+2)(\alpha+1)}\left\{\frac{\rho_0{}^2}{(\alpha+1)\alpha}+k_1\right\}x_1{}^2+\frac{1}{(\alpha+3)(\alpha+2)}\right.$$
$$\left.\left\{\frac{\rho_0}{(\alpha+2)(\alpha+1)}\overbrace{\frac{\rho_0{}^2}{(\alpha+1)\alpha}+k_1}+\frac{\rho_0k_1}{(\alpha+1)\alpha}\right\}x_1{}^3.+\quad.\quad.\quad.\right] \quad (17)$$

where C_0 is an arbitrary constant.

$$R_1=C_0x_1\left[1+\frac{\rho_0}{2\cdot1}x_1+\frac{1}{3\cdot2}\left(\frac{\rho_0{}^2}{2\cdot1}+k_1\right)x_1{}^2+\frac{1}{4\cdot3}\left\{\frac{\rho_0}{3\cdot2}\left(\frac{\rho_0{}^2}{2\cdot1}+k_1\right)\right.\right.$$
$$\left.\left.+\frac{\rho_0k_1}{2\cdot1}\right\}x_1{}^3+\ldots\right]\quad.\quad.\quad.\quad.\quad(18)$$

Thus the complete solution of (1) and (2) can be found by combining the results given in (A), (9·A) and (11), multiplied by e^{at},

i.e.,
$$u=\frac{e^{at}}{\rho_1}P_2\left(x_1P_1-\frac{dp_1}{dx_1}\right)$$
$$v=e^{at}\left[\frac{1}{\rho_1}\frac{\partial P_2}{\partial y}\left(\frac{1}{k}x_1\frac{dP_1}{dx_0}-p_1\right)+R\frac{\partial S}{\partial z}\right]$$
$$w=e^{at}\left[\frac{1}{\rho_1}\frac{\partial P_2}{\partial z}\left(\frac{1}{k}x_1\frac{dP_1}{dx_1}-p_1\right)-R\frac{\partial S}{\partial y}\right]$$

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MOTION OF AN INCOMPRESSBILE FLUID WITH VARYING COEFFICIENT OF VISCOSITY, GIVEN BY $\mu=\mu_0+\varepsilon_1 x$. FOR POSITIVE VALUES OF x. PART III.

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In the previous two papers, *viz.*, Parts (I)[1] and (II), we have discussed the cases where ε_1 was considered small. There when u, v and w also were small we retained terms up to the first order of small quantities. In Part (I) we discussed the motion having neglected the inertia terms, whereas in Part (II) the inertia terms were retained up to the first order of small quantities. Here we have dwelt on the case when ε_1 is not necessarily small. The discussion has been divided into two parts, namely, (A) and (B). In this case also we shall assume that u, v and w are small, so that here too we shall neglect small quantities of the second order.

In Part (A) we have discarded inertia terms from the equations of motion whereas in (B) they have been retained.

At the origin the value of μ is μ_0 and we shall further suppose that μ is constant and equal to zero for negative values of x and the density of the liquid has been taken constant throughout the motion in Case (B).

From Paper (I) we find that in the absence of extraneous forces, our equations of motion are

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+(\mu_0+\varepsilon_1 x)\ \nabla^2 u+\varepsilon_1\left(2\frac{\partial u}{\partial x}\right)$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+(\mu_0+\varepsilon_1 x)\ \nabla^2 v+\varepsilon_1\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+(\mu_0+\varepsilon_1 x)\ \nabla^2 w+\varepsilon_1\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

with
$$\theta\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

We shall consider two cases, when u, v and w are small.

Firstly we shall neglect the inertia terms and secondly they will be retained.

(A) Neglecting the inertia terms we get the equations of motion to be

$$\left.\begin{aligned}(\mu_0+\varepsilon_1 x)\nabla^2 u+\varepsilon_1\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right)&=\frac{\partial p}{\partial x}\\(\mu_0+\varepsilon_1 x)\nabla^2 v+\varepsilon_1\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)&=\frac{\partial p}{\partial y}\\(\mu_0+\varepsilon_1 x)\nabla^2 w+\varepsilon_1\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)&=\frac{\partial p}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We use the transformation

$$\mu_0+\varepsilon_1 x=\varepsilon_1 x_1$$

so that $$\delta x=\delta x_1$$

whereby (1) and (2) are changed to

$$\left.\begin{aligned}\varepsilon_1 x_1\nabla^2 u+\varepsilon_1\left(\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_1}\right)&=\frac{\partial p}{\partial x_1}\\\varepsilon_1 x_1\nabla^2 v+\varepsilon_1\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x_1}\right)&=\frac{\partial p}{\partial y}\\\varepsilon_1 x_1\nabla^2 w+\varepsilon_1\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x_1}\right)&=\frac{\partial p}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1\cdot 1)$$

with $$\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2\cdot 1)$$

where $$\nabla^2\equiv\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

In (1·1) differentiating the first equation with respect to x_1, the second with respect to y and the third with respect to z, and adding and then using (2·1) we get

$$2\varepsilon_1\nabla^2 u=\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking the first of (1·1) we get

$$2\,\varepsilon_1\,x_1\nabla^2 u+4\varepsilon_1\frac{\partial u}{\partial x_1}=2\frac{\partial p}{\partial x_1}$$

Using (3) we get

$$4\varepsilon_1\frac{\partial u}{\partial x_1}=2\frac{\partial p}{\partial x_1}-x_1\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again $$\frac{\partial^2 p}{\partial x_1^2}=\varepsilon_1\,x_1\frac{\partial}{\partial x_1}\nabla^2 u+\varepsilon_1\nabla^2 u+2\varepsilon_1\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)$$

or $$4\frac{\partial^2 p}{\partial x_1^2}=4\varepsilon_1\,x_1\frac{\partial}{\partial x_1}\nabla^2 u+4\varepsilon_1\nabla^2 u+2\cdot 4\varepsilon_1\frac{\partial}{\partial x_1}\left(\frac{du}{\partial x_1}\right)$$

Using(4) we get $4\dfrac{\partial^2 p}{\partial x_1^2}=x_1\nabla^2\left(2\dfrac{\partial p}{\partial x_1}-x_1\nabla^2 p\right)+2\nabla^2 p+2\dfrac{\partial}{\partial x_1}\left(2\dfrac{\partial p}{\partial x_1}-x_1\nabla^2 p\right)$

or $\quad 4\dfrac{\partial^2 p}{\partial x_1^2}=2x_1\dfrac{\partial}{\partial x_1}\nabla^2 p-x_1\left\{x_1\nabla^2(\nabla^2 p)+2\dfrac{\partial}{\partial x_1}\nabla^2 p\right\}$

$$+2\nabla^2 p+4\frac{\partial^2 p}{\partial x_1^2}-2\left(\nabla^2 p+x_1\frac{\partial}{\partial x_1}\nabla^2 p\right)$$

or $\quad 0=2x_1\dfrac{\partial}{\partial x_1}\nabla^2 p-x_1^2\nabla^2(\nabla^2 p)-2x_1\dfrac{\partial}{\partial x_1}\nabla^2 p+2\nabla^2 p-2\nabla^2 p$

$$-2x_1\frac{\partial}{\partial x_1}\nabla^2 p$$

or $\quad x_1\nabla^2(\nabla^2 p)+2\dfrac{\partial}{\partial x_1}\nabla^2 p=0 \quad \ldots \ldots \quad (5)$

Put $\nabla^2 p=\mathrm{P} \quad \ldots \ldots \quad (6)$

so that

$$x_1\nabla^2\mathrm{P}+2\frac{\partial\mathrm{P}}{\partial x_1}=0 \quad \ldots \ldots \quad (7)$$

Thus (5), (6) and (7) combined will give us p.

Now let $\mathrm{P}=\mathrm{P}_1\mathrm{P}_2$ where P_1 is a function of x_1 only and P_2 that of y and z only.

Substituting the value of P in (7) we get

$$x_1\left(\mathrm{P}_2\frac{d^2\mathrm{P}_1}{dx_1^2}+\mathrm{P}_1\nabla_1^2\mathrm{P}_2\right)+2\mathrm{P}_2\frac{d\mathrm{P}_1}{dx_1}=0,\quad \text{where } \nabla_1^2\equiv\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

or $\quad \dfrac{1}{\mathrm{P}_1}\dfrac{d^2\mathrm{P}_1}{dx_1^2}+\dfrac{2}{x_1}\dfrac{1}{\mathrm{P}_1}\dfrac{d\mathrm{P}_1}{dx_1}+\dfrac{1}{\mathrm{P}_2}\nabla_1^2\mathrm{P}_2=0.$

This can be satisfied if we write

$$\frac{1}{\mathrm{P}_1}\frac{d^2\mathrm{P}_1}{dx_1^2}+\frac{2}{x_1}\frac{1}{\mathrm{P}_1}\frac{d\mathrm{P}_1}{dx_1}=k \text{ and } \nabla_1^2\mathrm{P}_2+k\mathrm{P}_2=0. \quad \ldots \quad (8)$$

or $\quad x_1\dfrac{d^2\mathrm{P}_1}{dx_1^2}+2\dfrac{d\mathrm{P}_1}{dx_1}-kx_1\mathrm{P}_1=0 \quad \ldots \ldots \quad (9)$

Let $\mathrm{D}_1\equiv x_1\dfrac{d^2}{dx_1^2}+2\dfrac{d}{dx_1}-kx_1$, so that our equation becomes

$$\mathrm{D}_1\mathrm{P}_1=0. \quad \ldots \ldots \quad (9\cdot 1)$$

Following the method of Frobenius,[2] we construct an expression

$$\bar{z}=c_0 x_1^{\alpha}+c_1 x_1^{\alpha+1}+c_2 x_1^{\alpha+2}+\ldots+c_n x_1^{\alpha+n}+\ldots$$

We have $x_1\mathrm{D}_1\bar{z}=\Sigma c_0\left[\{\alpha(\alpha-1)+2\alpha\}x_1^{\alpha}-kx_1^{\alpha+2}\right]$

$$=c_0\alpha(\alpha+1)x_1^{\alpha}-c_0 k x_1^{\alpha+2}$$

$$+c_1(\alpha+1)(\alpha+2)x_1^{\alpha+1}-c_1 k x_1^{\alpha+3}$$

$$+c_2(\alpha+2)(\alpha+3)x_1^{\alpha+2}-c_2k\,x_1^{\alpha+4}$$
$$+\ \ldots\ldots\ldots\ldots\ldots$$
$$+\ \ldots\ldots\ldots\ldots\ldots$$
$$+c_{n-2}(\alpha+n-2)(\alpha+n-1)x_1^{\alpha+n-2}-c_{n-2}\,k.x_1^{\alpha+n}$$
$$+c_{n-1}(\alpha+n-1)(\alpha+n)x_1^{\alpha+n-1}-c_{n-1}\,k\,x_1^{\alpha+n+1}$$
$$+c_n(\alpha+n)(\alpha+n+1)x_1^{\alpha+n}-c_n\,k\,x_1^{\alpha+n+2}$$
$$+\ \ldots\ldots\ldots\ldots\ldots$$
$$+\ \ldots\ldots\ldots\ldots\ldots$$

or
$$x_1 D_1\,\overline{Z}=c_0\,\alpha(\alpha+1)\,x_1^{\alpha} \qquad (10)$$
provided
$$(\alpha+1)(\alpha+2)\,c_1=0 \qquad (11)$$
$$(\alpha+n)(\alpha+n+1)\,c_n-c_{n-2}.k=0,\ \text{for } n\geqq 2 \qquad (12)$$

From (11) and (12) we find

$$c_1=c_3=c_5=\cdots\cdots=0 \qquad (13)$$

Our indical equation is given by

$$\alpha(\alpha+1)=0$$

whence $\alpha=0$ or -1 . . . . (14)

Now from (12)

$$\frac{c_n\,x_1^2}{c_{n-2}}=\frac{kx_1^2}{(\alpha+n)(\alpha+n+1)}, \qquad \text{for } n\geqq 2$$
$$\longrightarrow 0 \text{ as } n\to\infty$$

Hence the series under consideration will be absolutely and uniformly convergent for all values of x_1. Since the difference of the two values of α differ by an integer, the two solutions of (9) will be given by

$$P_1'=\left[\,\bar{z}\,\right]_{\alpha=0} \quad \text{and} \quad P_1''=\left[\frac{\partial\bar{z}}{\partial\alpha}\right]_{\alpha=-1}$$

From (12)

$$c_n=\frac{k}{(\alpha+n)(\alpha+n+1)}\,c_{n-2}, \qquad \text{for } n\geqq 2$$

and from (13)
$$c_1=c_3=c_5=\cdots\cdots=0$$

so that
$$c_2=\frac{k}{(\alpha+2)(\alpha+3)}c_0$$
$$c_4=\frac{k}{(\alpha+4)(\alpha+5)}\,c_2=\frac{k}{(\alpha+4)(\alpha+5)}\cdot\frac{k}{(\alpha+2)(\alpha+3)}\,c_0$$
$$c_6=\frac{k}{(\alpha+6)(\alpha+7)}\quad\frac{k}{(\alpha+4)(\alpha+5)}\cdot\frac{k}{(\alpha+2)(\alpha+3)}\,c_0$$

etc.

Hence

$$\bar{z}=c_0\, x_1^{\alpha}\left[1+\frac{k}{(\alpha+2)(\alpha+3)}x_1^{2}+\frac{k^2}{(\alpha+4)(\alpha+5)(\alpha+2)(\alpha+3)}x_1^{4}\right.$$
$$+\frac{k^3}{(\alpha+6)(\alpha+7)(\alpha+4)(\alpha+5)(\alpha+2)(\alpha+3)}x_1^{6}$$
$$\left.+\;.\;.\;.\;.\;.\;.\;.\right],$$

where c_0 is an arbitrary constant. . . . . (15)

$$\frac{\partial\bar{z}}{\partial\alpha}=\log x_1\,.\,\bar{z}-c_0\, x_1^{\alpha+2}\left[\frac{kx_1^{2}}{(\alpha+2)(\alpha+3)}\left(\frac{1}{\alpha+2}+\frac{1}{\alpha+3}\right)\right.$$
$$+\frac{k^2x_1^{4}}{(\alpha+4)(\alpha+5)(\alpha+2)(\alpha+3)}\left(\frac{1}{\alpha+4}+\frac{1}{\alpha+5}+\frac{1}{\alpha+2}+\frac{1}{\alpha+3}\right)$$
$$+\frac{k^3\,x_1^{6}}{(\alpha+6)(\alpha+7)(\alpha+4)(\alpha+5)(\alpha+2)(\alpha+3)}\left(\frac{1}{\alpha+6}+\frac{1}{\alpha+7}+\frac{1}{\alpha+4}+\frac{1}{\alpha+5}\right.$$
$$\left.\left.+\frac{1}{\alpha+2}+\frac{1}{\alpha+3}\right)+\ldots\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\right] \qquad (15\cdot1)$$

Thus

$$\left\{\begin{aligned}P_1'&=c_0\left[1+\frac{k}{2.3}x_1^{2}+\frac{k^2}{2.3.4.5}x_1^{4}+\frac{k^3}{2.3.4.5.6.7}x_1^{6}+\ldots\right],\\ P_1''&=c_0\frac{\log x_1}{x_1}\left[1+\frac{k}{1.2}\,.\,x_1^{2}+\frac{k^2}{1.2.3.4}\,x_1^{4}+\frac{k^3}{1.2.3.4.5.6}x_1^{6}+\ldots\right]\end{aligned}\right.$$
$$-c_0\, x_1\left[\frac{kx_1^{2}}{2!}(1+\tfrac{1}{2})+\frac{k^2x_1^{4}}{4!}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}\right)+\frac{k^3x_1^{6}}{6!}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}\right)+\ldots\right],$$

i.e.,
$$\left\{\begin{aligned}P_1'&=c_0\left[1+\frac{kx_1^{2}}{3!}+\frac{k^2x_1^{4}}{5!}+\frac{k^3x_1^{6}}{7!}+\;.\;.\;.\;.\right], \qquad (15\cdot2)\\ P_1''&=\frac{c_0\log x_1}{x_1}\left[1+\frac{k^2x_1^{2}}{2!}+\frac{k^2x_1^{4}}{4!}+\frac{k^3x_1^{6}}{6!}+\ldots\right]\end{aligned}\right.$$
$$-c_0x_1\left[\begin{aligned}&\frac{kx_1^{2}}{2!}(1+\tfrac{1}{2})+\frac{k^2x_1^{4}}{4!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})\\ &+\frac{k^2x_1^{6}}{6!}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6})+\ldots\end{aligned}\right],\;.\;.\;.\;.\;.\qquad (15\cdot3)$$

For ρ_2 we consider (8).

$$\nabla_1^2P_2+kP_2-0$$

This gives[3]

$$\left.\begin{aligned}P_2&=B_s'\;\;J_s(\beta_1\omega)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\,s\phi,\text{ if } k=\beta_1^2\\ \text{or}\qquad &B_m''\;\;I_m(\beta_2\omega)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\,m\phi,\text{ if } k=-\beta_2^2\end{aligned}\right]\;.\;.\;.\;.\;(16)$$

From (6) we get

$$\nabla^2 p = P = P_1 P_2$$

Put $p = P_2 p_1$, where p_1 is a function of x_1 only

whence $$p_1 = c_1^1 e^{\sqrt{k}\, x_1} + c_2^1 e^{-\sqrt{k}\, x_1} + \frac{1}{D^2 - k} P_1 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\therefore \; p = \Sigma\, p_n + p_1 P_2. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where p_n is a solid harmonic of degree n.

We write $\varepsilon_1 u = u_0$, $\varepsilon_1 v = v_0$ and $\varepsilon_1 w = w_0$ so that we get from (1·1)

$$\left.\begin{aligned} \frac{\partial p}{\partial x_1} &= x_1 \nabla^2 u_0 + \left(\frac{\partial u_0}{\partial x_1} + \frac{\partial u_0}{\partial x_1}\right) \\ \frac{\partial p}{\partial y} &= x_1 \nabla^2 v_0 + \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x_1}\right) \\ \frac{\partial p}{\partial z} &= x_1 \nabla^2 w_0 + \left(\frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial x_1}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1{\cdot}2)$$

with $$\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2{\cdot}2)$$

We can write (4) as

$$4\frac{\partial u_0}{\partial x_1} = 2\frac{\partial p}{\partial x_1} - x_1 \nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}1)$$

or $\quad 4u_0 = P_2(2p_1 - \int x_1 P_1 dx_1) + cf(y_1 z)$, where c is an arbitrary constant.

Putting $c=0$, we get a particular solution to be given by

$$4u_0 = P_2(2p_1 - \int x_1 P_1 dx_1) \quad . \quad . \quad . \quad . \quad . \quad (4{\cdot}2)$$

The solutions of (1·2) and (2·2) are given by

$$\left.\begin{aligned} 4u_0 &= P_2(2p_1 - \int x_1 P_1 dx_1) \\ v_0 &= \frac{\partial P_2}{\partial y} \cdot v_1 \\ w_0 &= \frac{\partial P_2}{\partial z} \cdot v_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where v_1 is to be determined from

$$4kv_1 = \frac{2dp_1}{dx_1} - x_1 P_1 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

For $$4\left(\frac{\partial u_0}{\partial x_1} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z}\right) = P_2\left(2\frac{dp_1}{dx_1} - x_1 P_1\right) + 4v_1 \nabla_1^2 P_2$$

$$= P_2\left(2\frac{dp_1}{dx_1} - x_1 P_1 - 4kv_1\right), \qquad \text{by (8)}$$

$$= 0 \qquad \text{by (20)}$$

Again

$$4\left(x_1\nabla^2 v_0+\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x_1}\right)$$

$$=4x_1\nabla^2\left(v_1\frac{\partial P_2}{\partial y}\right)+\left(2p_1-\int x_1 P_1 dx_1\right)\frac{\partial P_2}{\partial y}+4\frac{\partial}{\partial x_1}\left(v_1\frac{\partial P_2}{\partial y}\right)$$

$$=4x_1\left\{\frac{\partial P_2}{\partial y}\frac{d^2}{dx_1^2}\left(v_1\right)-k\frac{\partial P_2}{\partial y}v_1\right\}+\frac{\partial P_2}{\partial y}\left(2p_1-\int x_1 P_1 dx_1\right)+4\frac{\partial P_2}{\partial y}\frac{dv_1}{dx_1}$$

$$=\frac{\partial P_2}{\partial y}\left[\frac{x_1}{k}\frac{d^2}{dx_1^2}\left(2\frac{dp_1}{dx_1}-x_1P_1\right)-4kx_1v_1+2p_1-\int x_1P_1dx_1+\frac{1}{k}\frac{d}{dx_1}\left(2\frac{dp_1}{dx_1}-x_1P_1\right)\right]$$

$$=\frac{\partial P_2}{\partial y}\left[\frac{x_1}{k}\frac{d}{dx_1}\left(2\frac{d^2p_1}{dx_1^2}-P_1-x_1\frac{dP_1}{dx_1}\right)-x_1\left(2\frac{dp_1}{dx_1}-x_1P_1\right)+2p_1-\int x_1P_1dx_1\right.$$
$$\left.+\frac{1}{k}\left(2\frac{d^2p_1}{dx_1^2}-P_1-x_1\frac{dP_1}{dx_1}\right)\right]$$

$$=\frac{\partial P_2}{\partial y}\left[\frac{x_1}{k}\frac{d}{dx_1}\left(P_1+2kp_1-x_1\frac{dP_1}{dx_1}\right)-2x_1\frac{dp_1}{dx_1}+x_1^2P_1+2p_1-\int x_1P_1dx_1\right.$$
$$\left.+\frac{1}{k}\left(P_1+2kp_1-x_1\frac{dP_1}{dx_1}\right)\right]$$

$$=\frac{1}{k}\frac{\partial P_2}{\partial y}\left[x_1\frac{dP_1}{dx_1}+2kx_1\frac{dp_1}{dx_1}-x_1\frac{dP_1}{dx_1}-x_1\frac{d^2P_1}{dx_1^2}-2kx_1\frac{dp_1}{dx_1}+kx_1^2P_1\right.$$
$$\left.+2kp_1-k\int x_1P_1dx_1+\left(P_1+2kp_1-x_1\frac{dP_1}{dx_1}\right)\right]$$

$$=\frac{1}{k}\frac{\partial P_2}{\partial y}\left[-x_1\left(x_1\frac{d^2P_1}{dx_1^2}+\frac{dp_1}{dx_1}-kx_1P_1\right)+4kp_1+P_1-k\int x_1P_1dx_1\right]$$

$$=\frac{1}{k}\frac{\partial P_2}{\partial y}\left[x_1\frac{dP_1}{dx_1}+P_1-k\int x_1P_1dx_1\right]+4p_1\frac{\partial P_2}{\partial y}\qquad\text{by (9)}$$

$$=4\frac{\partial p}{\partial y}+\frac{1}{k}\frac{\partial P_2}{\partial y}\left[x_1\frac{dp_1}{dx_1}+P_1-k\int x_1P_1dx_1\right]$$

From (9) we get

$$x_1\frac{d^2P_1}{dx_1^2}+2\frac{dP_1}{dx_1}-kx_1P_1=0$$

or
$$\frac{d}{dx_1}\left(x_1\frac{dP_1}{dx_1}+P_1\right)-k\int x_1P_1dx_1=\text{a constant};$$

there being no additive function of y and z as P_1 is purely a function of x_1. Hence choosing the constant to be zero, we find a particular solution that satisfies

$$x_1\nabla^2v_0+\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x_1}\right)=\frac{\partial p}{\partial y}.$$

Similarly we get w_0.

Now we shall consider the solutions of

$$\left.\begin{aligned} x_1\nabla^2 u^1{}_0 + \left(\frac{\partial u^1{}_0}{\partial x_1} + \frac{\partial u^1{}_0}{\partial x_1}\right) &= 0 \\ x_1\nabla^2 v_0{}^1 + \left(\frac{\partial u^1{}_0}{\partial y} + \frac{\partial v^1{}_0}{\partial x_1}\right) &= 0 \\ x_1\nabla^2 w_0{}^1 + \left(\frac{\partial u^1{}_0}{\partial z} + \frac{\partial w^1{}_0}{\partial x_1}\right) &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21)$$

with

$$\frac{\partial u^1{}_0}{\partial x_1} + \frac{\partial v^1{}_0}{\partial y} + \frac{\partial w^1{}_0}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2\cdot2)$$

Differentiating the first equation of (21) with respect to x_1, the second with respect to y and the third with respect to z, and adding and then using (2·2) we get

$$\nabla^2 u^1{}_0 = 0, \text{ so that } \frac{\partial u^1{}_0}{\partial x_1} = 0$$

$\therefore$ $u^1{}_0 = \mathrm{S}$, a function of y and z only, where $\nabla_1{}^2\mathrm{S} = 0$.

Hence
$$\frac{\partial v^1{}_0}{\partial y} + \frac{\partial w^1{}_0}{\partial z} = 0$$

The solutions of (21) and (2·2) may be given by

$$\left.\begin{aligned} u^1{}_0 &= \mathrm{S} \\ v^1{}_0 &= \frac{\partial \mathrm{S}}{\partial y} \cdot \xi_0 \\ w^1{}_0 &= \frac{\partial \mathrm{S}}{\partial z} \cdot \xi_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where ξ_0, a function of x_1 only, is to be found from

$$x_1 \frac{d^2\xi_0}{dx^2{}_1} + \frac{d\xi_0}{dx_1} = -1 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or
$$\frac{d}{dx_1}\left(x_1 \frac{d\xi_0}{dx_1}\right) = -1$$

or
$$x_1 \frac{d\xi_0}{dx_1} = -x_1 + \delta_1, \text{ where } \delta_1 \text{ is an arbitrary constant}$$

or
$$d\xi_0 = \frac{\delta_1 - x_1}{x_1}\, dx_1$$

or
$$\xi_0 = \delta_1 \log x_1 - x_1 + \delta_2, \quad \delta_2 \text{ being another arbitrary constant.} \quad (24)$$

Hence a complete solution of (1) and (2) can be found by combining the results of (19) and (22).

Thus
$$\left.\begin{aligned} u &= \frac{1}{4\varepsilon_1} P_2 \left((2p_1 - \int x_1 P_1 dx_1 \right) + \frac{S}{\varepsilon_1} \\ v &= \frac{1}{\varepsilon_1}\frac{\partial P_2}{\partial y} v_1 + \frac{1}{\varepsilon_1}\frac{\partial S}{\partial y}\xi_0 \\ w &= \frac{1}{\varepsilon_1}\frac{\partial P_2}{\partial y} v_1 + \frac{1}{\varepsilon_1}\frac{\partial S}{\partial z}\xi_0 \end{aligned}\right\}$$

(B) Retaining the inertia terms up to the first order of small quantities, we get the equations of motion to be

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu_0+\varepsilon_1 x}{\rho}\nabla^2 u + \frac{\varepsilon_1}{\rho}\left(2\frac{\partial u}{\partial x}\right) \\ \frac{\partial v}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu_0+\varepsilon_1 x}{\rho}\nabla^2 v + \frac{\varepsilon_1}{\rho}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \\ \frac{\partial w}{\partial t} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu_0+\varepsilon_1 x}{\rho}\nabla^2 w + \frac{\varepsilon_1}{\rho}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \end{aligned}\right\} \quad . \quad . \quad (25)$$

with
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (26)$$

where ρ is a constant.

Supposing u, v and w, all, varying as $e^{\alpha_1 t}$, where α_1 is a constant, and putting $\rho\alpha_1 = \rho_1$ and $\mu_0 + \varepsilon_1 x = \varepsilon_1 x_1$ so that $\delta x = \delta x_1$, we find

$$\left.\begin{aligned} u &= -\frac{1}{\rho_1}\frac{\partial p}{\partial x_1} + \frac{\varepsilon_1}{\rho_1} x_1 \nabla^2 u + \frac{\varepsilon_1}{\rho_1}\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_1}\right) \\ v &= -\frac{1}{\rho_1}\frac{\partial p}{\partial y} + \frac{\varepsilon_1}{\rho_1} x_1 \nabla^2 v + \frac{\varepsilon_1}{\rho_1}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x_1}\right) \\ w &= -\frac{1}{\rho_1}\frac{\partial p}{\partial z} + \frac{\varepsilon_1}{\rho_1} x_1 \nabla^2 w + \frac{\varepsilon_1}{\rho_1}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x_1}\right) \end{aligned}\right\} \quad . \quad . \quad (25\cdot1)$$

with
$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad (26\cdot1)$$

where ∇^2 now means $\dfrac{\partial^2}{\partial x_1{}^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

In physical problems, due to dissipative forces if the velocity continually diminishes, α_1 must be negative.

Differentiating the equation of (25·1) with respect to x_1, the second with respect to y, and the third with respect to z, and adding we have

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho_1}\nabla^2 p + \frac{\varepsilon_1}{\rho_1} x_1 \nabla^2\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\varepsilon_1}{\rho_1}\nabla^2 u$$
$$+ \frac{\varepsilon_1}{\rho_1}\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial m}{\partial z}\right) + \frac{\varepsilon_1}{\rho}\nabla^2 u$$

or $$0 = -\frac{1}{\rho_1}\nabla^2 p + \frac{2\varepsilon_1}{\rho_1}\nabla^2 u, \qquad \text{by } (26\cdot1)$$

or $$2\varepsilon_1 \nabla^2 u = \nabla^2 p \qquad \ldots \ldots \quad (27)$$

Again from (25·1)

$$u = -\frac{1}{\rho_1}\frac{\partial p}{\partial x_1} + \frac{\varepsilon_1}{\rho_1}x_1\nabla^2 u + \frac{\varepsilon_1}{\rho_1}\left(2\frac{\partial u}{\partial x_1}\right)$$

or $$2\varepsilon_1 u = -\frac{2\varepsilon_1}{\rho_1}\frac{\partial p}{\partial x_1} + \frac{\varepsilon_1}{\rho_1}x_1\nabla^2 p + \frac{4\varepsilon_1^2}{\rho_1}\frac{\partial u}{\partial x_1}$$

or $$2\varepsilon_1\nabla^2 u = -\frac{2\varepsilon_1}{\rho_1}\frac{\partial}{\partial x_1}\nabla^2 p + \frac{\varepsilon_1}{\rho_1}\nabla^2(x_1\nabla^2 p) + \frac{4\varepsilon_1^2}{\rho_1}\frac{\partial}{\partial x_1}\nabla^2 u$$

or $$\nabla^2 p = -\frac{2\varepsilon_1}{\rho_1}\frac{\partial}{\partial x_1}\nabla^2 p + \frac{\varepsilon_1}{\rho_1}\left\{x_1\nabla^2(\nabla^2 p) + 2\frac{\partial}{\partial x_1}\nabla^2 p\right\} + \frac{2\varepsilon_1}{\rho_1}\frac{\partial}{\partial x_1}\nabla^2 p$$

or $$\nabla^2 p = \frac{\varepsilon_1}{\rho_1}\left\{x_1\nabla^2(\nabla^2 p) + 2\frac{\partial}{\partial x_1}\nabla^2 p\right\}$$

Substituting $\nabla^2 p = \mathrm{P}$ and $\rho_1/\varepsilon_1 = \rho_0$, we get the differential equation for determining P to be given by

$$x_1\nabla^2\mathrm{P} + 2\frac{\partial \mathrm{P}}{\partial x_1} - \rho_0\mathrm{P} = 0\,. \qquad \ldots \ldots \quad (28)$$

We put $\mathrm{P} = \mathrm{P}_1\mathrm{P}_2$ as in Case (A)

so that $$\mathrm{P}_2\frac{d^2\mathrm{P}_1}{dx_1^2} + \mathrm{P}_1\nabla_1^2\mathrm{P}_2 + \frac{2}{x_1}\mathrm{P}_2\frac{d\mathrm{P}_1}{dx_1} - \frac{\rho_0}{x_1}\mathrm{P}_1\mathrm{P}_2 = 0,$$

where $$\nabla_1^3 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

or $$\frac{1}{\mathrm{P}_1}\frac{d^2\mathrm{P}_1}{dx_1^2} + \frac{2}{x_1}\cdot\frac{1}{\mathrm{P}_1}\cdot\frac{d\mathrm{P}_1}{dx_1} - \frac{\rho_0}{x_1} + \frac{1}{\mathrm{P}_2}\nabla_1^2\mathrm{P}_2 = 0$$

This can be satisfied if we write

$$\left.\begin{aligned} &\frac{1}{\mathrm{P}_1}\frac{d^2\rho_1}{dx_1^2} + \frac{2}{x_1}\frac{1}{\mathrm{P}_1}\frac{d\mathrm{P}_1}{ux_1} - \frac{\rho_0}{x_1} = k \\ \text{and}\quad &\frac{1}{\mathrm{P}_2}\nabla_1^2\mathrm{P}_2 + k = 0 \end{aligned}\right\} \quad k \text{ being an arbitrary constant.}$$

or $$x_1\frac{d^2\rho_1}{dx_1^2} + 2\frac{d\mathrm{P}}{dx_1} - (\rho_0 + kx_1)\,\rho_1 = 0 \qquad \ldots \ldots \quad (29)$$

and $$\nabla_1^2\mathrm{P}_2 + k\mathrm{P}_2 = 0 \qquad \ldots \ldots \quad (30)$$

First we consider the solution of (29).

$$\text{Let } D_1 \equiv x_1 \frac{d^2}{dx_1{}^2} + 2\frac{d}{dx_1} - (\rho_0 + kx_1).$$

In finding the solution, we shall follow the method of Frobenius. Constructing an expression

$$\overline{V} = a_0 x_1{}^{\alpha} + a_1 x_1{}^{\alpha+1} + a_2 x_1{}^{\alpha+2} + \ldots\ldots + a_n x_1{}^{\alpha+n} + \ldots\ldots$$

We have

$$\begin{aligned} x_1 D_1 \overline{V} = {} & \Sigma a_0 \left[\{\alpha(\alpha-1) + 2\alpha\} x_1{}^{\alpha} - \rho_0 x_1{}^{\alpha+1} - kx_1{}^{\alpha+2}\right] \\ = {} & a_0\{\alpha(\alpha+1)\, x_1{}^{\alpha} - \rho_0 x_1{}^{\alpha+1} - kx_1{}^{\alpha+2}\} \\ & + a_1\{(\alpha+1)(\alpha+2)\, x_1{}^{\alpha+1} - \rho_0 x_1{}^{\alpha+2} - kx_1{}^{\alpha+3}\} \\ & + a_2\{(\alpha+2)(\alpha+3)\, x_1{}^{\alpha+2} - \rho_0 x_1{}^{\alpha+3} - kx_1{}^{\alpha+4}\} \\ & + \ldots\ldots\ldots \\ & + \ldots\ldots\ldots \\ & + a_{n-2}\{(\alpha+n-2)(\alpha+n-1)\, x_1{}^{\alpha+n-2} - \rho_0 x_1{}^{\alpha+n-1} - kx_1{}^{\alpha+n}\} \\ & + a_{n-1}\{(\alpha+n-1)(\alpha+n)\, x_1{}^{\alpha+n-1} - \rho_0 x_1{}^{\alpha+n} - kx_1{}^{\alpha+n+1}\} \\ & + a_n\{(\alpha+n)(\alpha+n+a1)\, x_1{}^{\alpha+n} - \rho_0 x^{\alpha+n+1} - kx_1{}^{\alpha+n+2}\} \\ & + \ldots\ldots\ldots \end{aligned}$$

or $\qquad x_1 D_1 \overline{V} = a_0 \alpha (\alpha+1) x_1{}^{\alpha} \quad \ldots\ldots\ldots \quad (31)$

provided $a_1 (\alpha+1)(\alpha+2) - \rho_0 a_0 = 0 \quad \ldots\ldots\ldots \quad (32)$

and $\qquad a_n (\alpha+n)(\alpha+n+1) - \rho_0 a_{n-1} - k a_{n-2} = 0, \qquad$ for $n \geq 2 \quad . \quad (33)$

Our indical equation is given by

$$\alpha(\alpha+1) = 0$$

whence $\qquad \alpha = 0 \text{ or } -1 \quad \ldots\ldots \quad (34)$

Now from (33)

$$\frac{a_n}{a_{n-2}} = \frac{k}{(\alpha+a)(\alpha+n+1)} + \frac{\rho_0}{(\alpha+n)(\alpha+n+1)} \frac{a_{n-1}}{a_{n-2}}$$

Proceeding to limits as $n \longrightarrow \infty$, we find that

$$\frac{a_n}{a_{n-1}} \longrightarrow 0$$

i.e. $\qquad \underset{n \longrightarrow \infty}{\mathrm{Lt}} \frac{a_n x_1}{a_{n-1}} = 0$

Hence the series thus found will be absolutely and uniformly convergent for all values of x_1. Since the difference of the two values of α differ by an integer, the two solutions of (29) will be given by

$$P_1' = \Big[V \Big]_{\alpha=0} \quad \text{and} \quad \rho_1'' = \left[\frac{\partial \overline{V}}{\partial \alpha} \right]_{\alpha=-1}$$

by suitably modifying the arbitrary constant.

From (32)

$$a_1 = \frac{\rho_0}{(\alpha+1)(\alpha+2)} a_0$$

From (33)

$$a_n = \frac{1}{(\alpha+n)(\alpha+n+1)} \Big(k\, a_{n-2} + \rho_0\, a_{n-1} \Big), \qquad \text{for } n \geq 2.$$

$$a_2 = \frac{1}{(\alpha+2)(\alpha+3)} (k a_0 + \rho_0 a_1)$$

$$= \frac{1}{(\alpha+2)(\alpha+3)} \left\{ k + \frac{\rho_0^2}{(\alpha+1)(\alpha+2)} \right\} a_0$$

$$a_3 = \frac{1}{(\alpha+3)(\alpha+4)} \Big(k a_1 + \rho_0\, a_2 \Big)$$

$$= \frac{1}{(\alpha+3)(\alpha+4)} \left\{ \frac{k\rho_0}{(\alpha+1)(\alpha+2)} + \frac{\rho_0}{(\alpha+2)(\alpha+3)}\, k + \frac{\rho_0^2}{(\alpha+1)(\alpha+2)} \right\} a_0,$$

and so on.

$$\text{Hence } \overline{V} = a_0 x_1^{\alpha} \Bigg[1 + \frac{\rho_0}{(\alpha+1)(\alpha+2)} x_1 + \frac{1}{(\alpha+2)(\alpha+3)} \left\{ k + \frac{\rho_0^2}{(\alpha+1)(\alpha+2)} \right\} x_1^2 + \frac{1}{(\alpha+3)(\alpha+4)} \left\{ \frac{k\rho_0}{(\alpha+1)(\alpha+2)} + \frac{\rho_0}{(\alpha+2)(\alpha+3)}\, k + \frac{\rho_0^2}{(\alpha+1)(\alpha+2)} \right\} x_1^3 + \cdots \Bigg], \qquad \ldots \ldots (35)$$

where a_0 is an arbitrary constant.

$$\text{Thus } P'_1 = a_0 \Bigg[1 + \frac{\rho_0}{1.2} x_1 + \frac{1}{2.3} \left(k + \frac{\rho_0^2}{1.2} \right) x_1^2 + \frac{1}{3.4} \left\{ \frac{k\rho_0}{1.2} + \frac{\rho_0}{2.3} \left(k + \frac{\rho_0^2}{1.2} \right) \right\} x_1^2 + \cdots \Bigg] \qquad \ldots \ldots (36)$$

To find P_1'', we have to modify the arbitrary constant and bring in a new one, b_0, where $a_0 = b_0\,(\alpha+1)$, in order to avoid zero factors in the denominators of the terms.

Again P_2 and p can be found as in Case (A).

We now write down the equations of motion in the following form :—

$$\left.\begin{aligned} \rho_0 u &= -\frac{1}{\varepsilon_1}\frac{\partial p}{\partial x_1} + x_1\nabla^2 u + \left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_1}\right) \\ \rho_0 v &= -\frac{1}{\varepsilon_1}\frac{\partial p}{\partial y} + x_1\nabla^2 v + \left(\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial y}\right) \\ \rho_0 w &= -\frac{1}{\varepsilon_1}\frac{\partial p}{\partial z} + x_1\nabla^2 w + \left(\frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial z}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (25\cdot 2)$$

with $$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (26\cdot 1)$$

We take the second equation of (25·2)

$$\rho_0 v = -\frac{1}{\varepsilon_1}\frac{\partial p}{\partial y} + x_1\nabla^2 v + \left(\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial y}\right)$$

or $$\rho_0\nabla^2 v = -\frac{1}{\varepsilon_1}\frac{\partial}{\partial y}\nabla^2 p + x_1\nabla^2(\Delta^2 v) + 2\frac{\partial}{\partial x_1}\nabla^2 v + \frac{\partial}{\partial x_1}\nabla^2 v + \frac{\partial}{\partial y}\nabla^2 u$$

Putting $\nabla^2 v = V = V_1 \dfrac{\partial P_2}{\partial y}$, where V_1 is a function of x_1 only

We get

$$\rho_0 V = -\frac{1}{\varepsilon_1}\frac{\partial P}{\partial y} + x_1\nabla^2 V + 3\frac{\partial V}{\partial x_1} + \frac{1}{2\varepsilon_1}\frac{\partial P}{\partial y}, \text{ by (27), and where } \nabla^2 p = P$$

or $$\frac{\partial P_2}{\partial y}\left[x_1\left(\frac{d^2 V_1}{dx_1^2} - kV_1\right) + 3\frac{dV_1}{dx_1} - \rho_0 V_1\right] = \frac{1}{2\varepsilon_1}P_1\frac{\partial P_2}{\partial y} \text{ by (30)}$$

or $$x_1\frac{d^2 V_1}{dx_1^2} + 3\frac{dV_1}{dx_1} - (\rho_0 + kx_1)V_1 = \frac{1}{2\varepsilon_1}P_1 \quad \ldots\ldots\ldots\ldots \quad (37)$$

Now $\nabla^2 v = \dfrac{\partial P_2}{\partial y}\,.\,V_1$

Let $v = v_1\dfrac{\partial \rho_2}{\partial y}$, where v_1 is a function of x_1 only.

This v_1 will be given by

$$\frac{d^2 v_1}{dx_1^2} - kv_1 = V_1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

wherby $$v_1 = b_1 e^{\sqrt{kx_1}} + b_2 e^{-\sqrt{kx_1}} + \frac{1}{D^2 - k}V_1 \quad \ldots\ldots\ldots\ldots \quad (39)$$

where b_1 and b_2 are two arbitrary constants.

Similarly if we proceed with the third equation of (25.2) as we have done in the case of the second we shall find that

$$w = v_1\frac{\partial P_2}{\partial z}$$

will satisfy the third equation.

From (27)

$$\nabla^2 u = \frac{1}{2\varepsilon_1} \nabla^2 p = \frac{P_2}{2\varepsilon_1} P_1$$

Let $u = \frac{P_2}{2\varepsilon_1} u_1$, where u_1 is a function of x_1 only,

so that $\frac{d^2 u_1}{dx_1^2} - k u_1 = P_1$

i.e., $u_1 = p_1$ . . . . (40)

$\therefore \quad u = \frac{P_2}{2\varepsilon_1} u_1 +$ spherical harmonic terms.

Here we shall neglect spherical harmonic terms to suit our purpose, *viz.*, to have a particular solution of the problem.

So in order that

$$\left.\begin{aligned} u &= \frac{P_2}{2\varepsilon_1} u_1 \\ v &= \frac{\partial P_2}{\partial y} v_1 \\ w &= \frac{\partial P_2}{\partial z} w_1 \end{aligned}\right\} \qquad (41)$$

may be solutions of (25·2) and (26·1)

$$\frac{1}{2\varepsilon_1}\frac{du_1}{dx_1} - k v_1 \text{ must be equal to zero}$$

i.e., $k v_1 = \frac{1}{2\varepsilon_1}\frac{du_1}{dx_1}$ must satisfy (38)

i.e.,

$$2\varepsilon_1 k V_1 = \frac{d^3 u_1}{dx_1^2} - k\frac{du_1}{dx_1} = \frac{dP_1}{dx_1} \qquad (42)$$

must satisfy (37).

Now substituting the expression for kV_1 in the left-hand side of (37)

we get

$$\frac{1}{2\varepsilon_1 k}\left[x_1 \frac{d^3 P_1}{dx_1^3} + 3\frac{d^2 P_1}{dx_1^2} - (\rho_0 + k x_1)\frac{dP_1}{dx_1}\right]$$

Again differentiating (29) with respect to x_1 we get

$$x_1 \frac{d^3 P_1}{dx_1^3} + 3\frac{d^2 P_1}{dx_1^2} - (\rho_0 + k x_1)\frac{dP_1}{dx_1} = kP_1$$

Hence

$$\frac{1}{2\varepsilon_1 k}\left[x_1 \frac{d^3 P_1}{dx_1^3} + 3\frac{d^2 P_1}{dx_1^2} - (\rho_0 + k x_1)\frac{dP_1}{dx_1}\right] = \frac{P_1}{2\varepsilon_1}$$

Therefore (37) is satisfied.

Thus

$$\left.\begin{aligned} 2\varepsilon_1 u &= p_1 \mathrm{P}_2 \\ 2\varepsilon_1 v &= \frac{1}{k}\frac{\partial \mathrm{P}_2}{\partial y}\cdot\frac{dp_1}{dx_1} \\ 2\varepsilon_1 w &= \frac{1}{k}\frac{\partial \mathrm{P}_2}{\partial z}\cdot\frac{dp_1}{dx_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

We shall, now, consider the solution of

$$\left.\begin{aligned} \rho_0 u' &= x_1 \nabla^2 u' + \left(\frac{\partial u'}{\partial x_1} + \frac{\partial u'}{\partial x_1}\right) \\ \rho_0 v' &= x_1 \nabla^2 v' + \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x_1}\right) \\ \rho_0 w' &= x_1 \nabla^2 w' + \left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x_1}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

with

$$\frac{\partial u'}{\partial x_1} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Differentiating the first equation of (44) with respect to x_1, the second with respect to y, and the third with respect to z, and adding and then using (45) we get

$$\nabla^2 u' = 0$$

so that

$$\rho_0 u' = 2\frac{\partial u'}{\partial x_1}$$

or

$$u' = \mathrm{S}\, e^{\frac{\rho_0 x_1}{2}}, \text{ where S is a function of } y \text{ and } z \text{ only} \quad . \quad . \quad . \quad (46)$$

Again

$$\rho_0\left(\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) = x_1 \nabla^2\left(\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right) + \nabla_1^2 u' + \frac{\partial}{\partial x_1}\left(\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)$$

or

$$-\frac{\rho_0^2}{2}\mathrm{S}e^{\rho_0\frac{x_1}{2}} = -x_1\frac{\rho_0}{2}\nabla^2\left(\mathrm{S}e^{\frac{\rho_0 x_1}{2}}\right) + e^{\frac{\rho_0 x_1}{2}}\nabla_1^2\mathrm{S} - \frac{\rho_0^2}{4}\mathrm{S}e^{\rho_0\frac{x_1}{2}}, \text{ by (46)}$$

or

$$-\frac{\rho_0^2}{2}\mathrm{S}e^{\rho_0\frac{x_1}{2}} = -x_1\frac{\rho_0}{2}\left[\mathrm{S}.\frac{\rho_0^2}{4}e^{\rho_0\frac{x_1}{2}} + e^{\rho_0\frac{x_1}{2}}\nabla_1^2\mathrm{S}\right] + e^{\rho_0\frac{x_1}{2}}\nabla_1^2\mathrm{S} - \frac{\rho_0^2}{4}\mathrm{S}e^{\frac{\rho_0 x_1}{2}}$$

or

$$-\frac{\rho_0^2}{2}\mathrm{S} = -\frac{\rho_0^3}{8}x_1\mathrm{S} - \frac{\rho_0}{2}x_1\nabla_1^2\mathrm{S} + \nabla_1^2\mathrm{S} - \frac{\rho_0^2}{4}.\mathrm{S}$$

or

$$\left(\frac{\rho_0}{2}x_1 - 1\right)\nabla_1^2\mathrm{S} = \frac{\rho_0^2}{4}\mathrm{S}\left(1 - \rho_0\frac{x_1}{2}\right)$$

or

$$\Delta_1^2\mathrm{S} + \frac{\rho_0^2}{4}\mathrm{S} = 0$$

or

$$(\nabla_1^2 + k_1^2)\,\mathrm{S} = 0, \text{ where } k_1 = \frac{\rho_0}{2}$$

$\therefore$

$$\mathrm{S} = \mathrm{A}_l\, \mathrm{J}_l\,(k_1\omega)\begin{matrix}\cos\\ \sin\end{matrix}\Big\}\, l\phi,^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Thus we find that

$$\left.\begin{aligned} u' &= \mathrm{S}e^{\rho_0 \frac{x_1}{2}} \\ v' &= \frac{2}{\rho_0}\frac{\partial \mathrm{S}}{\partial y} e^{\rho_0 \frac{x_1}{2}} \\ w' &= \frac{2}{\rho_0}\frac{\partial \mathrm{S}}{\partial z} e^{\rho_0 \frac{x_1}{2}} \end{aligned}\right\} \quad \cdots\cdots \quad (48)$$

The complete solution of (25) and (26) will be obtained by combining the results of (43) and (48).

Hence

$$\left.\begin{aligned} e^{-a_1 t}\, u &= \frac{1}{2\varepsilon_1} p_1\, \mathrm{P}_2 && + \mathrm{S}e^{\rho_0 \frac{x_1}{2}} \\ e^{-a_1 t}\, v &= \frac{1}{2\varepsilon_1}\frac{1}{k}\frac{\partial \mathrm{P}_2}{\partial y}\frac{dp_1}{dx_1} && + \frac{2}{\rho_0}\frac{\partial \mathrm{S}}{\partial y} e^{\rho_0 \frac{x_1}{2}} \\ e^{-a_1 t}\, w &= \frac{1}{2\varepsilon_1}\frac{1}{k}\frac{\partial \mathrm{P}_2}{\partial z}\frac{dp_1}{dx_1} && + \frac{2}{\rho_0}\frac{\partial \mathrm{S}}{\partial z} e^{\rho_0 \frac{x_1}{2}} \end{aligned}\right\} \quad \cdots \quad (49)$$

My best thanks are due to Prof. A. C. Banerji for his kind help in finding the solution of the problem.

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ADIABATIC PULSATIONS OF THE CEPHEID VARIABLE

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SUMMARY

Any model in which the density varies inversely as a power of the distance from the centre has been shown to have singularities at the centre. Instability of radial oscillations has been shown for linear and quadratic laws of density. The period of small oscillation of the homogeneous star has been calculated and found to accord very closely with observation.

Profs. Eddington,[8] Edgar,[13] Schwartzschild[16] and others have considered the modes of small radial oscillations of the standard polytrope (of index 3). Eddington's calculation of the period of oscillation[12] comes short about by a factor 2, pointing to a lower central condensation in actual Cepheids than in the standard polytrope. It is evidently quite important to consider as many laws of variation of density as possible, and, to this end, Dr. Sterne[17] has considered small oscillations of the following three stellar models :

(1) the star of uniform density,

(2) the star in which the density varies inversely as the square of the distance from the centre, and

(3) the star in which nearly all the mass is contained in a particle at the centre and in the rest of the star the density varies inversely as the square of the distance from the centre.

Very recently Prof. A. C. Banerji[1] has given an entirely new and interesting theory of the origin of the solar system, based on the instability of large radial oscillations of the following models :

(1) the star of uniform density, and

(2) the star with a small, homogeneous core, and the density in the annulus varying inversely as the pth power of the distance from the centre, where p is any positive integer excluding 1 and 3.

Banerji's investigations throw an important light on the density condensation in Cepheid variables, and following his method the author has shown in another paper[19] that if the density vary inversely as a power of the distance from the centre, radial oscillations are possible only for the aforementioned Sterne's model (2). The analysis supports the inference drawn from

observation that the Cepheid variables are approximately homogeneous configurations.[4, 15]

In the present paper it has been shown that Sterne's model (2) and, in fact, any model in which the density varies inversely as a power of the distance from the centre, would have singularities at the centre. Linear and quadratic forms have therefore been assumed for the law of density, and the oscillations in these cases have been shown to be unstable. The validity of the solutions for the standard model has been questioned, and it is surmised that any law of density proceeding in power series of the distance from the centre would lead to instability of radial oscillations, so that radial oscillations can only be possible for a homogeneous star. The author is attempting a rigorous proof of this, which, if arrived at, would be presented in a subsequent paper. As an exercise, the period of small oscillation of the homogeneous star has been calculated, and is found to accord much better with observation than Eddington's theoretical estimate for the standard model.[12]

We first show that any law for which the density varies inversely as some power of the distance from the centre would lead to singularities at the centre. We will consider the following cases :

Case (1).

$$\rho=\frac{k}{r}. \qquad (1)$$

where ρ is the density at a point distant r from the centre, and k is a constant.

We will suppose the sphere to be of radius R, having a small, homogeneous (to be ultimately evanescent) core of radius

$$a=\mu R \qquad (2)$$

and the law of density in the annulus given by (1).

Let $\bar{\rho}$ be the mean density of the sphere. Then we have

$$\tfrac{4}{3}\pi a^3.\frac{k}{a}+\int_a^R 4\pi r^2.\frac{k}{r}dr = \tfrac{4}{3}\pi R^3 \bar{\rho}, \qquad (3)$$

whence we have

$$k\longrightarrow\tfrac{2}{3}R\bar{\rho}, \qquad \text{as } \mu\longrightarrow 0 . \qquad (4)$$

Now, mass inside radius a

$$=\tfrac{4}{3}\pi a^3.\frac{k}{a}=\tfrac{4}{3}\pi k\mu^2R^2 \qquad (5)$$

Hence the mass of the core becomes evanescent with the core.

Now, as $\mu\longrightarrow 0$, mass inside radius r $(r\neq 0)$

$$\longrightarrow 2\pi kr^2\longrightarrow\frac{4\pi}{3}R\bar{\rho}r^2. \qquad (6)$$

Hence the gravity, g, at r $(r \neq 0) \longrightarrow \frac{4\pi G}{3} R \bar{\rho}$ = constant.

But the value of the central gravity is zero, as the central mass is evanescent.

Again, the pressure P_r at r $(r \neq 0)$

$$=g\int_r^R \rho dr \longrightarrow \frac{8\pi}{9} GR^2 \bar{\rho}^2 \log\frac{R}{r}, \text{ as } \mu \longrightarrow 0. \qquad (7)$$

From (7) we have $P_r \longrightarrow \infty$, as $r \longrightarrow 0$.

The model, therefore, is not a physically possible configuration, as there is a discontinuity in the value of gravity at the centre, and the central pressure tends to infinity.

Case (2).

The same model as in Case (1), with the law (1) replaced by

$$\rho = \frac{k}{r^2} \qquad (8)$$

We have for the mass in the sphere

$$\tfrac{4}{3}\pi a^3 . \frac{k}{a^2} + \int_a^R 4\pi r^2 . \frac{k}{r^2} dr = \tfrac{4}{3}\pi R^3 \bar{\rho} \qquad (9)$$

From (9) we have

$$k \longrightarrow \tfrac{1}{3} R^2 \bar{\rho}, \quad \text{as } \mu \longrightarrow 0. \qquad (10)$$

Hence the mass inside the core of radius a

$$= \tfrac{4}{3}\pi a^3 . \frac{k}{a^2} = \tfrac{4}{3}\pi k\mu R \longrightarrow 0, \qquad \text{as } \mu \longrightarrow 0. \qquad (11)$$

The mass inside radius r $(r \neq 0)$

$$\longrightarrow \frac{4\pi}{3} R^2 \bar{\rho} r \qquad (12)$$

From (12) we have for the gravity at r $(r \neq 0)$,

$$g \longrightarrow \frac{4\pi G}{3} . \frac{R^2 \bar{\rho}}{r} \qquad (13)$$

From (13) we have $g \longrightarrow \infty$, as $r \longrightarrow 0$.

But the value of the gravity at the centre is zero, as the central mass is evanescent.

Again, we have for the pressure at r ($r \neq 0$),

$$P_r = \int_r^R g\rho dr \longrightarrow \frac{2\pi G}{9} R^4 \bar{\rho}^2 \left(\frac{1}{r^2} - \frac{1}{R^2}\right), \text{ as } \mu \longrightarrow 0. \quad . \quad . \quad . \quad (14)$$

From (14) we have $P_r \longrightarrow \infty$, as $r \longrightarrow 0$.

Hence, the model is not a physically possible configuration for the same reasons as in Case (1).

Case (3).

$$\rho = \frac{k}{r^p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where p is an integer greater than 2.

It can be easily shown in this case that the central mass tends to infinity as $\mu \to 0$, so that the model is not a physically possible configuration.

We thus see that if the density vary inversely as some power of the distance from the centre, the star would have singularities at the centre. Thisc an be avoided by assuming the law of density to hold up to the surface of a central, homogeneous core of small but finite radius. Instability of this model for large radial oscillations has been shown by Banerji[1] and for small oscillations by the author.[19] Another way to avoid the singularity would be to assume the law of density to be a power series of the distance of the point from the centre. The investigation in the general case has not yet been tractable. We will here consider the following two simple laws of density :

$$(a) \quad \rho = \rho_c \left(1 - \frac{r}{R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$(b) \quad \rho = \rho_c \left(1 - \frac{r^2}{R^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where ρ_c is the central density, R the radius of the star, and ρ the density at a point r. We assume the density to vanish on the surface.

The differential equation[9] for small adiabatic oscillations of a star has been obtained by Eddington in the form :

$$\frac{d^2 a_1}{d\xi_0^2} + \frac{4-\nu}{\xi_0} \frac{da_1}{d\xi_0} + \left\{\frac{n^2}{\gamma} \frac{\rho_0}{P_0} - \left(3 - \frac{4}{\gamma}\right) \frac{\nu}{\xi_0^2}\right\} a_1 = 0, \quad . \quad . \quad . \quad (18)$$

where ξ_0, ρ_0, P_0 and g_0 are the undisturbed values of the distance from the centre, density, pressure and gravity,

$$\nu = \frac{g_0 \rho_0 \xi_0}{P_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\xi - \xi_0 = \delta\xi = \xi_0 \xi_1 = \xi_0 a_1 \cos nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

with similar equations for the other variables,

$$\frac{\delta P}{P_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where γ is the effective ratio of the specific heats[10] (regarding the matter and enclosed radiation as one system), and

$$n = \frac{2\pi}{\Pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where Π is the period of the pulsation.

Changing to the independent variable

$$x = \frac{\xi_0}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

the equation (18) becomes

$$\frac{d^2 a_1}{dx^2} + \frac{4-\nu}{x}\frac{da_1}{dx} + \left\{R^2 \frac{n^2}{\gamma}\frac{\rho_0}{P_0} - \frac{\alpha\nu}{x^2}\right\} a_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where ν is given by (19), and

$$\alpha = 3 - \frac{4}{\gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We will now consider the following cases :

Case (a).

With the law of density as given in (16) we have

$$g_0 = \frac{G}{\xi_0^2}\int_0^{\xi_0} 4\pi\xi_0^2\rho_0\, d\xi_0 = 4\pi\, G\rho_c\, \xi_0\left(\frac{1}{3} - \frac{\xi_0}{4R}\right), \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$P_0 = \int_{\xi_0}^{R} g_0\rho_0 d\xi_0 = 4\pi G/\rho_c^2\left(\frac{5}{144}R^2 - \frac{\xi_0^2}{6} + \frac{7\,\xi_0^3}{36\;R} - \frac{\xi_0^4}{16R^2}\right), \quad . \quad . \quad (27)$$

from the value of g_0 in (26).

Hence, we have from (16), (19), (26) and (27)

$$\nu = \frac{12x^2(4-3x)}{(1-x)\,(5+10x-9x^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where x is given by (23).

Also, we have

$$\frac{R^2\rho_0}{P_0} = \frac{36}{\pi G\rho_c(1-x)\,(5+10x-9x^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Substituting from (28) and (29) in (24), we have

$$x(1-x)\,(5+10x-9x^2)\frac{d^2a_1}{dx^2}+(20+20x-124x^2+72x^3)\,\frac{da_1}{dx}$$
$$+\{f-12\alpha(4-3x)\}xa_1=0, \qquad (30)$$

where

$$f=\frac{36n^2}{\pi G\rho_c\gamma} \qquad (31)$$

The equation (30) has regular singularities[21] at $x=0$ and $x=1$.

Assume

$$a_1=x^q\sum_0^\infty b_\lambda x^\lambda \qquad (32)$$

as a series solution of (30).

The indicial equation[21] gives $q=0$ or -3. Retaining only the former value to avoid singularity at the centre, we have the recurrence formula :

$$(9\lambda^2+27\lambda-90+36\alpha)b_{\lambda-2}-\{19\lambda^2+67\lambda-86-(f-48\alpha)\}b_{\lambda-1}+(5\lambda^2+15\lambda)b_\lambda$$
$$+5(\lambda^2+5\lambda+4)b_{\lambda+1}=0 \qquad (33)$$

(33) may be put in the form

$$(9-19x'+5y'+5z')\lambda^2+(27-67x'+15y'+25z')\lambda+36\alpha-90$$
$$+(86+f-48\alpha)x'+20z'=0, \qquad (34)$$

where $\quad x'=b_{\lambda-1}/b_{\lambda-2},\ y'=b_\lambda/b_{\lambda-2}$, and $z'=b_{\lambda+1}/b_{\lambda-2} \qquad (35)$

Dividing (34) by λ^2, and proceeding to the limit as $\lambda\to\infty$, we have

$$9-19\mathrm{Lt}x'+5\mathrm{Lt}y'+5\mathrm{Lt}z'=0 \qquad (36)$$

Putting $\mathrm{Lt}x'=l$,

we have

$$\mathrm{Lt}\,y'=\mathrm{Lt}\,(b_\lambda/b_{\lambda-1}\cdot b_{\lambda-1}/b_{\lambda-2})=(\mathrm{Lt}\,x')^2=l^2 \qquad (37)$$

and

$$\mathrm{Lt}\,z'=l^3 \qquad (38)$$

Hence, we have from (36)

$$9-19l+5l^2+5l^3=0 \qquad (39)$$

Equation (39) has $l=1$ as one root, giving unit radius of convergence for the power series. We do not consider the other roots of (39), as it follows from the general theory[14] of linear differential equations having regular singularities that the power series (32) about the origin is convergent up to the next nearest singularity which is $x=1$.

We proceed to test the convergence of (32) for $x=1$.

As Lt $x'=l=1$, we have all the terms to be ultimately of the same sign, and

$$x'=1-\varepsilon, \qquad (40)$$

where[20]

$$\varepsilon=O\left(\frac{1}{\lambda^p}\right), \qquad (41)$$

p being positive.

Then we have from (35)

$$y'=(1-\varepsilon)^2=1-2\varepsilon \qquad (42)$$

and

$$z'=(1-\varepsilon)^3=1-3\varepsilon, \qquad (43)$$

to the first power of ε.

Substituting in (34) we obtain, equating to zero the terms of the highest order,

$$\varepsilon=\frac{16-12\alpha+f}{6\lambda^2} \qquad (44)$$

As α given by (25) lies[10] between 0 and ·6 and f in (31) is not negative, the numerator of the right-hand side of (44) is not zero, and we have

$$\varepsilon=O\left(\frac{1}{\lambda^2}\right). \qquad (45)$$

Hence, we have from (40) and (45)

$$b_\lambda/b_{\lambda+1}=1+\varepsilon=1+O\left(\frac{1}{\lambda^2}\right). \qquad (46)$$

We have from (46)

$$\underset{\lambda\to\infty}{\mathrm{Lt}}\ \lambda(b_\lambda/b_{\lambda+1}-1)=\underset{\lambda\to\infty}{\mathrm{Lt}}\ O\left(\frac{1}{\lambda}\right)=0. \qquad (47)$$

The series solution (32) is therefore divergent[2] for $x=1$.

By an extension of Abel's theorem[3] to series divergent on the circle of convergence, we have

$$\underset{x\to 1}{\mathrm{Lt}}\ \sum_0^\infty b_\lambda x^\lambda=\sum_0^\infty b_\lambda=\infty. \qquad (48)$$

Thus the amplitude of the oscillations increases without limit as we approach the surface of the star, and the model in question is therefore unstable for radial oscillations.

Case (*b*).

With the law of density as given in (17) we have

$$g_0=4\pi\rho_c G\left(\frac{\xi_0}{3}-\frac{\xi_0^3}{5R^2}\right), \qquad (49)$$

$$P_0=4\pi\rho_c^2 G\left(\frac{1}{15}R^2-\frac{\xi_0^2}{6}+\frac{2}{15}\frac{\xi_0^4}{R^2}-\frac{\xi_0^6}{30R^4}\right), \qquad (50)$$

$$\mu=\frac{g_0\rho_0\xi_0}{P_0}=\frac{2x^2(5-3x^2)}{(1-x^2)\,(2-x^2)}\,. \qquad (51)$$

and
$$R^2\frac{\rho_0}{P_0}=\frac{15}{2\pi\rho_c G(1-x^2)\,(2-x^2)} \qquad (52)$$

With these substitutions the equation (24) becomes

$$x(1-x^2)\,(2-x^2)\frac{d^2\xi_1}{dx^2}+(8-22x^2+10x^4)\frac{d\xi_1}{dx}$$
$$+\{f-2\alpha(5-3x^2)\}x\xi_1=0, \qquad (53)$$

where

$$f=\frac{15n^2}{2\pi\rho_c\gamma G} \qquad (54)$$

The equation (53) has regular singularities at the origin and at $x=1$. We do not consider $x=\sqrt{2}$ as it lies outside the star.

Assume the following series solution of (53):

$$\xi_1=x^q\sum_0^\infty c_\lambda x^\lambda\,. \qquad (55)$$

The indicial equation gives $q=0$ or -3. Taking, as before, the former value, we find that the odd coefficients vanish and for the even coefficients we have the following recurrence formula:

$$(\lambda^2+3\lambda-18+6\alpha)\,b_{\lambda-3}-(3\lambda^2+13\lambda-16-f+10\alpha)\,b_{\lambda-1}$$
$$+(2\lambda^2+10\lambda+8)\,b_{\lambda+1}=0, \qquad (56)$$

where λ is even.
(56) may be put in the form

$$(1-3x'+2y')\,\lambda^2+\ (3-13x'+10y')\lambda$$
$$+\{(16+f-10\alpha)\,x'+8y'+6\alpha-18\}=0, \qquad (57)$$

where
$$x'=b_{\lambda-1}/b_{\lambda-3} \text{ and } y'=b_{\lambda+1}/b_{\lambda-3}. \qquad (58)$$

Dividing by λ^2, and proceeding to the limit as $\lambda\to\infty$, we have

$$1-3\,\mathrm{Lt}x'+2\,\mathrm{Lt}y'=0. \qquad (59)$$

Putting
$$\mathrm{Lt}x'=l, \qquad (60)$$

we have
$$\mathrm{Lt}\,y'=l^2. \qquad (61)$$

Hence, we have from (59)

$$1-3l+2l^2=0. \qquad (62)$$

Equation (62) has $l=1$ as one root, giving unit radius of convergence for the power series (55). We do not consider the other root of (62), as the power

series (55) about the origin is convergent up to the next nearest singularity of (53), which is $x=1$.

We proceed to test the convergence of (55) for $x=1$.

As Lt $x'=l=1$, we have all the terms to be ultimately of the same sign, and

$$x'=1-\varepsilon, \qquad (63)$$

where

$$\varepsilon=O\left(\frac{1}{\lambda^p}\right), \qquad (64)$$

p being positive.

Then, we have from (58)

$$y'=(1-\varepsilon)^2=1-2\varepsilon, \qquad (65)$$

to the first power of ε.

Substituting in (57), we obtain

$$\varepsilon=\frac{f-4\alpha+6}{\lambda^2} \qquad (66)$$

As α given by (25) lies between 0 and ·6 and f in (54) is not negative, the numerator of the right-hand side of (66) is not zero, and we have

$$\varepsilon=O\left(\frac{1}{\lambda^2}\right) \qquad (67)$$

Hence, we have from (63) and (67)

$$b_{\lambda-1}/b_{\lambda+1}=1+\varepsilon=1+O\left(\frac{1}{\lambda^2}\right) \qquad (68)$$

We have from (68)

$$\mathrm{Lt}_{\lambda\to\infty}\frac{\lambda}{2}(b_{\lambda-1}/b_{\lambda+1}-1)=\mathrm{Lt}_{\lambda\to\infty}O\left(\frac{1}{\lambda}\right)=0 \qquad (69)$$

The series solution (55) is therefore divergent for $x=1$. By the extension of Abel's theorem to series divergent on the circle of convergence, we have

$$\mathrm{Lt}_{x\to 1}\sum_{0}^{\infty} c_\lambda x^\lambda = \sum_{0}^{\infty} c_\lambda = \infty \qquad (70)$$

Thus the amplitude of the oscillations increases without limit as we approach the surface of the star, and the model in question is therefore unstable for radial oscillations.

The investigation of the law of density as a power series in r becomes complicated and has not been attempted. However, in view of the present analysis, it becomes incumbent to establish the convergence of the series solution of the differential equation right up to the boundary of the star. Eddington's fundamental differential equation is (18). It is a differential equation of the second order having regular singularities at $\xi_0=0$ and $P_0=0$, that is, at the centre

and on the boundary of the star. From the general analysis[11] of linear differential equations having regular singularities, it is known that Frobenius' method gives series solutions convergent between two adjacent singularities. It is therefore necessary to establish the convergence of the series solution on the boundary of the star. This is the more so as Sterne[17] has obtained convergent solutions only for finite polynomials and the infinite series have become divergent on the boundary of the star. Similar results have been obtained by Banerji[1] and the author.[19]

The solutions obtained so far by quadrature of the standard model are unsatisfactory in this respect that the convergence or otherwise of the series solutions on the boundary has nowhere been tested. That the test is essential is evident from the fact that the amplitude on the surface becomes infinite for specific laws of density and there is no guarantee that it would remain finite for the standard model. This is apart from another objection to the current investigations, namely, that the square of the amplitude cannot be neglected. This has been fully elucidated by Banerji[1] in his paper.

The analysis so far has shown radial oscillations to be possible only for the homogeneous star. It is interesting to compare this with the following inferences drawn from observation :

Kopal[15] states, "If the hypotheses involved in the present state of the pulsation theory are correct, the density condensations of Cepheids must depend widely on their spectral class. With advancing spectral type the central condensations seem rapidly to diminish, and the δ Cephei F$-$5 stars seem to approach the limit of homogeneity." Chandrasekhar[4] states, "The Cepheids and the Cluster type variables which occur in the 'super-giant' region of the Hertzsprung-Russell diagram must be much less condensed towards the centre than the typical main series stars. The necessity for the Cepheids to be more homogeneous than 'ordinary stars' has often been emphasised by Eddington."

As a test of the hypothesis of homogeneity, we will calculate the period of the fundamental mode of small radial oscillations of the typical Cepheid, δ Cephei, assuming it to be homogeneous, and compare it with the observed period.

We have the following data[7] for δ Cephei :

Mass (Sun=1)	Radius (10^6 km.)	Period observed (days)	
8·8	20·2	5·37	. . (71)

Now Sterne[17] has proved that the period of the fundamental mode of oscillation of a homogeneous sphere of density ρ is given by

$$\frac{3n^2}{2\pi G\rho\gamma}=2\alpha, \qquad (72)$$

where G is the gravitational constant, and α, n and γ are given respectively by (25), (22) and (21).

Hence, the period

$$\Pi=\sqrt{\frac{3\pi}{G\gamma\alpha\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (73)$$

Now γ, according to Eddington,[11] is given by

$$\frac{\gamma-\frac{4}{3}}{\Gamma-\frac{4}{3}}=\frac{4-3\beta}{1+12\,(\Gamma-1)\,(1-\beta)/\beta}, \quad . \quad . \quad . \quad . \quad . \qquad (74)$$

where β is the ratio of gas to total pressure, and Γ is the ratio of specific heats for material gas alone.

γ is, therefore. not constant in the homogeneous star as β is not so. β, in fact, increases monotonically from the central value ·44 to the value 1 on the surface. This we show as follows :

We have the equations of gas and radiation pressure

$$\frac{\beta P}{\rho}=\frac{k}{\mu H}T \text{ and } (1-\beta)\,P=\tfrac{1}{3}aT^4, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75)$$

where k is the Boltzmann constant, a the Stefan constant, μ is the mean molecular weight, H is the mass of the proton, and T is the temperature. Eliminating T in (75), we have

$$\frac{\beta^4}{1-\beta}=\frac{3}{a}\left(\frac{k}{\mu H}\right)^4\frac{\rho^4}{P^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (76)$$

As ρ is constant for the homogeneous star and P decreases from centre to surface, it follows from (76) that β increases from centre to surface.

On the surface, $P=0$, and, therefore, from (76), $\beta=1$. We will calculate the central value of β. The central pressure P_c for the uniform star of mass M and radius R is given by[5]

$$P_c=\frac{3}{8\pi}\,\frac{GM^2}{R^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (77)$$

From the equations (76) and (77) we have β_c, the central value of β, given by

$$\frac{\beta_c{}^4}{1-\beta_c}=\frac{18}{a\pi}\left(\frac{k}{\mu H}\right)^4 G^3M^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (78)$$

Th equation (78) may be compared with Eddington's quartic equation[6] for the standard model. It shows that, as in the standard model, β_c depends only on the mass of the star and is independent of its radius.

Numerical solution of the quartic equation (78), with the help of the data (71) and a table of physical constants, gives

$$\beta_c=\cdot 44 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

for the homogeneous star.

As γ and α will vary with β, Sterne's solution[17] for the homogeneous star is not strictly valid. We may, however, as an approximation, assume an average value of β. The value of the period given by (73) will then depend on the assumed average.

We take $\Gamma = 1{\cdot}55$ as given by Fowler and Guggenheim[12] for the average Cepheid. For any assumed β, γ may be calculated from (74) and then Π from (73). Below we list the values of Π for three values of β, namely, the centre and surface values as well as the arithmetic mean of these.

	Centre	Surface	Average
β	·44	1	·72
Π (days)	10·4	5·65	8·0

As the observed period is that of the surface oscillation, clearly more weight should be attached to the surface value of β, and it is significant how closely the corresponding value of the period accords with the observational value of 5·37 days as given in (71). This should be compared with Eddington's computed value[12] of 3·57 days for the period of the standard model which is too low about by a factor 2. In fact, this suggests, as is well known, a lower central condensation for actual Cepheids than that in the standard model.[4]

Elsewhere the author[18] has shown that β is constant only for the standard model. A strictly rigorous discussion should, as has already been remarked, take account of the variation of β in the assumed model.

The above investigation has been carried out under the guidance of Prof. A. C. Banerji, to whom the author's most respectful thanks are due.

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THE STATIONARY CATHODE SPOT OF A LOW PRESSURE MERCURY ARC DISCHARGE

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The cathode spot of a low pressure Mercury arc wanders restlessly over the Mercury pool of the arc. A number of experimenters have observed, however, that it is possible to anchor the spot on a metal partly dipped in the Mercury pool ; the latest work on this subject is that of Warmoltz.[1] In the present work some characteristics of the phenomenon have been studied and data have been obtained to check the theory of the low pressure Mercury arc.

The Apparatus

The arc chamber, C, Fig. (1), is a flask of pyrex glass, 13 cms. in diameter. The cylindrical iron electrode A, 1·8 cms. long and 1·3 cms. in diameter, acts as the anode. A pool of clean and pure Mercury at the bottom of the flask serves as the cathode. The anode A is fixed in position in the flask with a thick Nickel lead screwed to A on one end and spot welded to a pyrex-Tungsten seal on the other. The entire length of the Nickel lead is covered with a pyrex capillary to shield it from the discharge. A gas-free Nickel rod, N, of desired diameter, sealed to the bottom of the flask, serves as an external lead to the Mercury pool. A small length, 1 to 2 mms. of this rod, is left projecting above the Mercury surface to act as an anchor. M is an additional movable Nickel electrode sealed to the main body of the apparatus through a high vacuum ground glass joint. The position of M was altered as desired without disturbing the vacuum in the arc by rotating this joint.

The apparatus was permanently connected to a single stage Gæde's all steel Mercury diffusion pump backed by a Hyvac. It was exhausted to a high vacuum and made air-tight before observations were taken. The gas pressure inside was measured with a Mcleod gauge. The pumps were kept continuously running during the observations taken.

The power for Mercury arc was drawn from a D.C. generator, G, which could give a current of 0—35 amperes at 100 volts. The arc voltage and current

were measured with D.C. moving coil instruments introduced in the circuit as shown in Fig. (1).

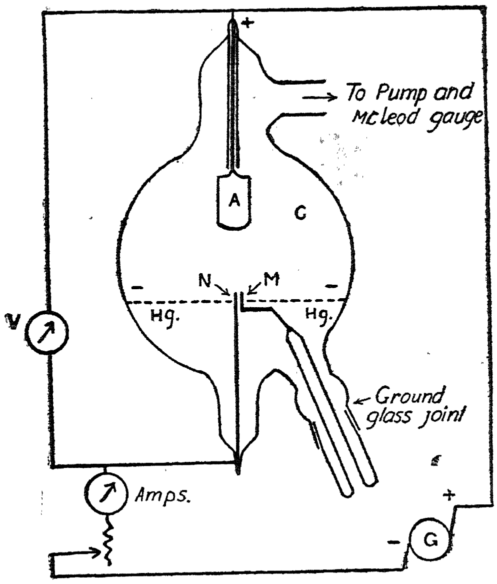

Fig. 1

The Results

As the arc is struck at a low current with a fresh Nickel rod the cathode spot starts racing over the Mercury surface and keeps dancing for a few minutes without touching the rod. It, however, gets attached to the rod at the common Ni-Hg boundary as the anchor is wetted by Mercury and remains permanently stationary there if the arc current is kept constant. The spot leaves its position temporarily if the arc current is increased, when its size also grows larger, but returns to the rod after moving restlessly over Mercury for a few seconds. This process continues till the size of the cathode spot, as a result of the increase in current, is large enough to encircle the rod completely. With further increase of arc current, the spot does not leave the rod unless the current reaches a critical value at which it gets detached from the rod. The spot now runs irregularly over the Mercury pool and the rod is as much favoured by it as any other point over the Mercury surface.

The behaviour of the arc is slightly different when a rod which has once been wetted with Mercury is used. The spot is attached to the rod immediately after the arc is struck at any current below its critical value. Moreover, it remains permanently stationary at the Ni-Hg boundary and grows continuously

in size without leaving the rod for a moment as the arc current is gradually increased to its critical limit.

The critical current is found to depend upon the thickness of the rod and rises linearly with its diameter. Besides, the current at which the spot encircles the rod completely is always observed to be a fraction of the critical current. This is true for rods of all diameters used.

Experiments were performed with anode at different heights from the Mercury pool and with rods of different metals. The critical current is found to be independent of the distance between the electrodes and the nature of the rod as well as its length projecting above the Mercury surface. The relation between the critical current and the diameter of a gas-free Nickel rod is shown in Fig. (2). Fig. (3) shows a current-voltage characteristic of the Mercury arc under a continuous run of about half an hour and is typical of a large number of curves taken.

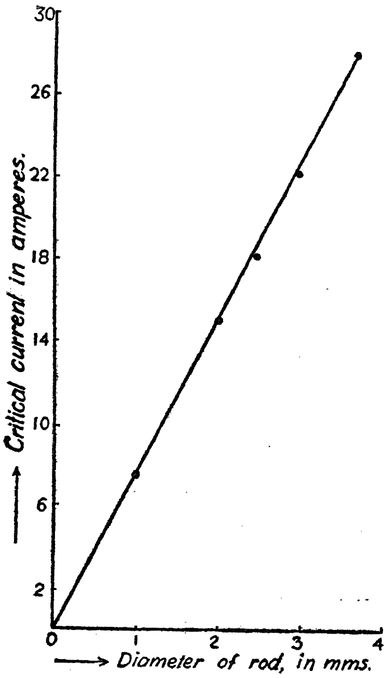

Fig. 2

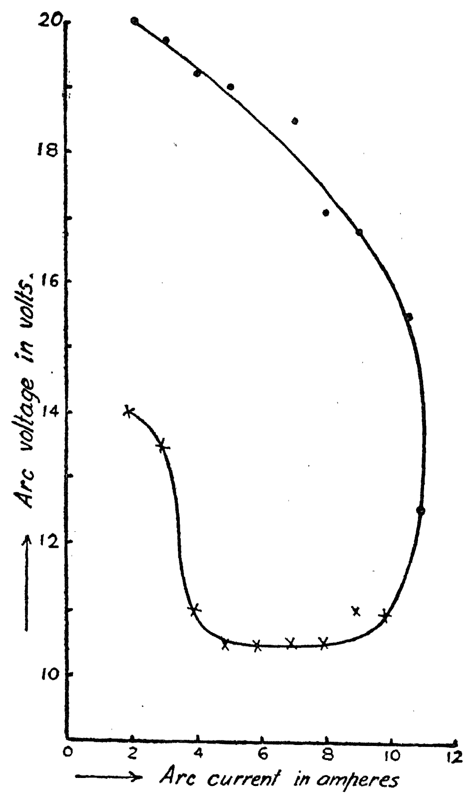

Fig. 3

Dotts - Current increasing. Crosses—Current decreasing.

Current Density in the Mercury Arc.—A nickel rod, 2 mms. in diameter and 4 mms. in length at the top and 5 mms. in diameter and 6 mms. in length at

the base, is partly immersed in the Mercury pool, the base resting 1—2 mms. deep below the Mercury surface. The cathode spot sticks to the Ni rod but as the current exceeds the critical value for a rod of 2 mms. diameter the spot makes a hole in Mercury and penetrates to the base which now acts as the anchor for the spot. The hole in Mercury becomes wider with increase of arc current till at a certain value the cathode spot is spread over the entire top of the base. The top is now exposed and made visible from outside, the volume of Mercury over it being pushed away to the sides. A simple calculation gives a current density of about 180 Amps./cm.^{2} in the Mercury arc.

Division of the cathode spot into components.—The movable Ni rod, M, is introduced into the apparatus at this stage. On striking the arc, with the two rods, M and N, in contact, the cathode spot sticks to one of them at the Ni-Hg boundary but encircles them both as the current is increased. The spot gets spilt into two components if the movable rod is removed sufficiently away from the other and each component sticks to one rod separately. The arc current is now easily increased to a value equal to the algebraic sum of the critical currents for the two rods without either of the spots getting detached from position. As the current exceeds this limit the spots leave the rods and combine into one which races over the Mercury surface for an indefinite period. The arc current is now decreased considerably below the total critical value without further sub-division of the spot which remains free all the time. It is re-attached, however, to one of the rods as the current approaches the corresponding limiting value and is steady in position with decreasing current till the arc dies out.

The spot behaves differently if the two rods are kept apart before striking the arc. It sticks to one of them for all currents below the corresponding critical value. As the current is increased beyond this limit the spot leaves the rod and is transferred as a whole to the other provided the latter is thicker and thus the critical current for it is higher, otherwise, it races over the Mercury pool as a free spot. The sub-division of the spot does not occur at all under these circumstances and the rods act independent of each other and the critical current of one is uninfluenced by the presence of the other. The character of the arc is also independent of the distance between the two rods.

Discussion

The mechanism of a low pressure Mercury arc is yet a subject of controversy as the factors governing the arc are not known with certainty. In building a theory of the Mercury arc the knowledge of (1) the current density at the cathode, (2) the cathode drop, (3) the width of cathode fall space and (4) the

temperature of the cathode spot is essential. The values obtained for these by various workers differ enormously.

The cathode dark space in the Hg arc is too small to be measured directly and can only be determined with the help of the space-charge equation which involves I and V, the current density and the cathode drop respectively. Lamar and Compton[2] have found the cathode drop to be about 10 volts and this is generally accepted as correct, though according to Stark[3] and his co-workers it should be 5·27 volts. Data for the current density are very few. Gunther-Schulze[4] finds its value to be as high as 4000 Amps./cm.^{2} while Thomsons[5] assume it to be 1000 Amps./cm.^{2} only.

Compton[6] estimates the temperature of the cathode spot to be not more than 200°c., while Thomsons[7] believe that the temperature of the spot is raised to about 3000°c. in a time as short as five millionths of a second.

In the present work an attempt has been made to determine the current density at the cathode of the arc. The cathode spot sticks to the base and probably to the central portion of the Ni rod as well. The effective area of the spot is consequently very slightly more than the area of the top of the base. We believe, however, that the error introduced due to the unknown active area of the rod is inappreciable.

According to Thomsons the emission of electrons from the cathode of a Mercury arc is thermionic but Langmuir[8] assumes it to be "Auto-electronic," caused by the intense electric field, of the order of a million volts per cm., existing at the cathode. The boiling point of Mercury is 357°c which is much below the temperatures at which effective thermionic emission occurs. One fails to understand how Mercury atoms can be heated to high temperatures of 3000–4000°c without boiling them off. The "field or auto-electronic" emission theory faces a similar set-back. The saturation current density on this theory is given by :—

$$j = 6{\cdot}2 \times 10^{-6} \frac{\mu^{\frac{1}{2}} \beta^2 F^2}{(\mu+\phi)\sqrt{\phi}} e^{-\frac{6{\cdot}8 \times 10^7 (\phi)^{\frac{3}{2}}}{\beta . F}}$$

where

(1) j = current density in Amps./cm.^{2}

(2) F is the electric field at the cathode and is of the order 10^6 volts per cm.

(3) $\phi = 5$ volts and represents the work function of Mercury.

(4) $\mu = 5$ volts and is a constant for Mercury.

(5) $B \sim 10$

This makes j of the order of 10^{-7} Amps/cm.^{2} which is vanishingly small as compared with the actual current density of the arc. It is hardly justified to

give preference to one theory over the other but the anchoring phenomenon of the cathode spot on a metal wetted with Mercury finds support on either.

The contact of a metal with a foreign surface reduces its work function and thereby increases its capacity for emitting the thermionic as well as the auto-electronic emission. The thin film of Mercury which probably gets deposited on the anchor thus becomes a better source of electrons for the arc than the Mercury pool, with the result, that the cathode spot sticks to the common boundary of the two metals. At critical and higher currents the evaporation of the Mercury film from the rod is probably very rapid and the spot remains detached from it till the next layer is formed. The fact that the spot leaves and returns to the rod repeatedly at high currents is in favour of the assumption.

The linear relation, Fig. (2), between the critical current and the diameter of the rod probably suggests that the size of the cathode spot is proportional to arc current. It also incidently shows that in the absence of a rod the spot would remain unstationary even at small arc currents.

The difference in the paths, Fig. (3), traced by the characteristic of the arc during a continuous run, on first increasing and then decreasing the arc current, is probably due to the variations in temperature of the arc.

The value of the current density found by us differ enormously from that of Gunther-Schulze. As it plays an extremely important rôle in the theory of the arc we have considered it desirable to obtain more data for it by improved methods. Experiments are now in progress in which the spot is studied photographically.

It is a pleasure to record our thanks to the Hon'ble late Sir Shah Mohd. Sulaiman, Vice-Chancellor, and Professor A. B. A. Haleem, Pro-Vice-Chancellor, for their interest in the work and the facilities provided for carrying it out.

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DEPENDENCE OF ACOUSTICAL IMPEDANCE OF SOUND-ABSORBING MATERIALS ON THEIR PHYSICAL PROPERTIES

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SUMMARY

An account is given in the paper of measurements, by an improved method, of the acoustic impedances of some common sound-absorbing materials like felt, treetax and insulating board. The experimental results obtained are discussed in relation to the theoretical work of Morse and others, which connects the impedance with the porosity and the flow resistance of the material. It is found that for soft materials like felt Morse's formula holds good, whereas for hard materials like insulating board the formula fails. Experimenting with an artificially constructed, hard, porous material, consisting of an assemblage of long vaccine tubes fixed to a brass disc, with their lengths normal to the plane of the disc, it is found that Morse's formula does not give even the proper order of magnitude of the impedance. When in this arrangement the interspaces between the capillary tubes are filled with beeswax and resin, i.e. when the tubes are embedded in this medium, so as to make the system rigid, it is found that the acoustic impedance as measured is just what should be expected from Rayleigh's formula for capillary tubes of finite length, showing that the experimental values are of the proper magnitude. The failure of Morse's formula in such hard materials, thus appears to be genuine. An attempt is made in the paper to explain this failure.

It has been shown by Morse and others[1] that the acoustical impedance of some systems can be interpreted on the basis of their physical properties. In the present paper an attempt is made to test the validity of the relation given by Morse and others, by applying it to sound absorbing materials like felt, treetax, insulating board and to an artificial sound absorber. For this purpose, measurements of the acoustical impedance in terms of its resistance and reactance components were made ; also the physical properties like porosity, flow resistance were experimentally determined in the laboratory and a comparison has been made of the calculated values with those found experimentally.

For determining the acoustical impedance, an electrical method was adopted, which consists in measuring the changes in the electrical impedance of a receiver unit, due to the presence of a sound absorbing material. The method is described in detail with its theory, full experimental details and typical calculations in previous papers.[2] For the present work the technique of the method was greatly improved and a few observations in the case of sound absorbing materials like felt, treetax and insulating board were repeated for greater accuracy.

To study the variation of resistance with wave-length of sound the results have been plotted in Fig. 1, for felt, treetax and insulating board. Dotted lines show the variation of reactance and solid lines of resistance.

In Fig. 1 dotted and solid lines marked (1) show the curves for felt. The resistance varies only from 1·2 to 2 ohms. The variation is very small. The reactance changes from −6 to −4 ohms and then as the frequency reaches about 1100 cycles it starts decreasing again. Since the experiment was not conducted beyond this frequency it cannot be said with certainty whether it will increase or decrease at frequencies higher than 1300 cycles.

Lines marked (2) show the variations in the case of treetax. Here the order of resistance is 11 ohms, much higher than that for felt. It varies from 10·7 to 11 ohms only, *i.e.*, it remains practically constant. Reactance varies from −3 ohms at 800 cycles to +3 ohms at 1300 cycles.

Lines marked (3) show the curves for an insulating board. The case of insulating board is quite different from any of these cases. The resistance does not remain constant but at first increases with frequency and then starts falling down after reaching a certain frequency. The changes of reactance are very rapid.

Beranek[3] gives results for a wide range of frequency and his apparatus being very sensitive and accurate, his results can be relied upon for comparison. Beranek's curves can be divided into three parts. In the first part the resistance remains practically constant and reactance increases with increasing frequency. In the second part, both reactance and resistance curves show a resonance effect and in the third part they practically become constant. Now comparing the present work with Beranek's it can be seen that in the case of felt only first part lies within our experimental range of frequency. The case of treetax is almost similar to that of felt. The case of insulating board is somewhat different from these two. Its second part lies within our experimental range of frequency. If we compare it with Beranek's curve for temacoustic we find that they are quite similar.

It can also be noted that as long as reactance remains negative the phase change remains less than 180° but as soon as it becomes positive the phase change increases beyond 180°.

It has been found that the absorbing power of treetax is more than that of felt or insulating board. Slight deviation of the results at places may be due to (1) the mounting conditions of the material, and (2) errors in the accurate determination of the wave-length of sound. The results for any material change with its mounting conditions, and as the material had to be taken out at every frequency and mounted again, some errors might have been introduced on that account. When, however, care is taken to avoid these errors, the method is

capable of giving the values of the resistance and reactance of a material and thus predict its behaviour at any frequency.

Physical properties of the sound-absorbing materials.—Morse, Bolts and Brown gave the equation for the value of acoustical impedance of the material in terms of its physical properties. The equation is

$$2\left(\frac{z}{\rho c}\right)\left(\frac{P}{m}\right)^{\frac{1}{2}} \cos\phi_r = (a+jb)\tanh\left[-j\sigma\,(a+jb)+\tfrac{1}{2}j\right] \quad . \quad . \quad (1)$$

$$a=\left\{2\left[1+\left(\frac{\gamma}{\sigma}\right)^2\right]^{\frac{1}{2}}+2\right\}^{\frac{1}{2}}$$

$$b=\left\{2\left[1+\left(\frac{\gamma}{\sigma}\right)^2\right]^{\frac{1}{2}}-2\right\}^{\frac{1}{2}}$$

$$\sigma=\left(\frac{L}{\lambda}\right)\left(mP\right)^{\frac{1}{2}}\cos\phi_r$$

$$\gamma=\left(\frac{rL}{2\pi\rho C}\right)\left(\frac{P}{m}\right)^{\frac{1}{2}}\cos\phi_r$$

L is the thickness of the material, λ the wave-length of sound in free air, ϕ_r is the angle of refraction of the wave in the material, given in terms of the angle of incidence ϕ_i by the equation.

$$\cos\phi_r=\left\{1-P^{-1}\left[m+j\left(\frac{r}{\rho\omega}\right)\right]^{-1}\sin^2\phi_i\right\}^{\frac{1}{2}}$$

In the present case the sound is incident normally so that $\sin\phi_i=0$ and hence $\cos\phi_r=1$.

P is the porosity of the material, defined as the ratio of the volume of free air in the material to the total volume of the material ; r is the effective resistivity or flow resistance of the material, defined as the ratio between the pressure drop per unit thickness and the volume flow of air through the material ; r has the dimensions c.c. per sec. per cm.^{2} of the surface ; $\frac{r}{\rho c}$ denotes the specific resistivity of the material, and has the dimensions cm.$^{-1}$. m denotes the ratio of the effective mass to the mass of an equal volume of air in the open, or the specific mass. As Kuhl, Meyer and Gemant have pointed out, m is usually greater than unity, since part of the pore material moves with the air and adds its inertia.

As a matter of fact none of these quantities will have at acoustic frequencies, exactly the same values as they have for steady state or geometrical measurements. Just as m is a constant giving the effective mass of the moving parts in the porous material, so is P a measure of the effective stiffness of the air and material combined. As defined, P is the ratio between the stiffness of the air alone to the acoustic stiffness of the air and material combined, and such a

quantity is liable to change with frequency. Although it is possible that its value at acoustic frequencies may be nearly equal to the value measured for steady flow.

Equation (1) given above is based on the assumption that the material is uniform and isotropic in porosity, etc., and that it has a rigid backing.

To find the acoustical impedance from the physical properties of the materials according to the above formula, the porosity[4] and flow resistance[5] of the materials were determined experimentally in the laboratory. The following table gives the values for all the three materials.

Table 1

Material	Thickness	Porosity	Resistivity
felt	1·1 cms.	·82	3·9
treetax	1·25 ,,	·62	30
insulating board ..	1·3 ,,	·25	175

Taking the case of felt for which $P = {\cdot}8$ and $\frac{r}{\rho c} = 4$ and using the known values of impedance, m is found to be about 10 gms. The thickness of the sample experimented upon was 1·1 cms. Putting these values of m, P, $\frac{r}{\rho c}$ and L in equation (1), the value of resistance and reactance terms of the acoustic impedance of felt were calculated at different frequencies. In Fig. 1, circles on curves show these computed values for resistance and reactance for felt.

Treetax was then studied in the same way. It has been found that as the value of $\frac{r}{\rho c}$ is large in this case, so the value of m can be very approximately determined. Taking it equal to 10, we find that the resistance is of the same order as found experimentally, but the reactance is much less. No other value of m gives better results, evidently it shows that the structure of treetax is not so simple as to be predicted by equation (1). The complexity of structure results in adding a mass reactance equal to $M\omega$ which is generally proportional to frequency.

Curves 2 in Fig. 1 refer to treetax. The crosses denote the experimental values and the circles show the computed values. In drawing curve 2′, a mass reactance equal to $M\omega$ has been added to the calculated reactance, which has not been done in drawing curve 2. The computed curves are now in agreement with the experimental curves. The small discrepancies observed may be due to the change of M with frequency, which we have neglected. The mass reactance required to explain the discrepancy observed above, may be due to the materials being less pervious at the surface than in the interior, or it may be

due to the skin effect or to the stiffness of the material against bending. Morse and others have also found while considering the results got by Beranek, that

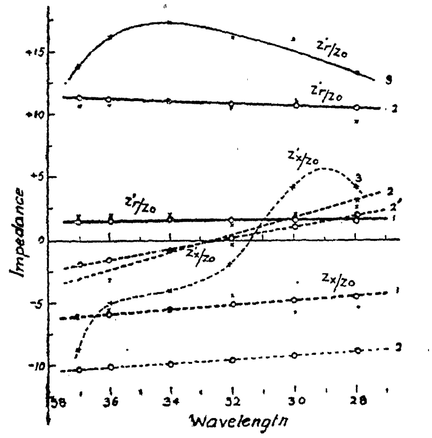

Fig. 1

The experimental values are denoted by crosses and the theoretical values by circles; the dotted lines represent the reactance and the solid lines the resistance. Curves 1 refer to felt, 2 to treetax, and 3 to insulating board.

in many cases the reactance computed theoretically is lower than the actual reactance.

In the case of insulating board it is found that for the experimental values of P and $\frac{r}{\rho c}$, no value of m gives resistance and reactance less than 50 and 45 ohms respectively whereas the experimental values are only 15 and 5 respectively. The experimental value of the flow resistance is 175 ; but it has been found by calculation that if the flow resistance is of the order of 40 the results come out of the same order as given by experiment. So much difference in $\frac{r}{\rho c}$ cannot be attributed to experimental errors. There are two possibilities in this case, (1) due to the complexity of structure a totally different relation in this case will predict the results ; or (2) the direct flow resistance method fails to give correct value of $\frac{r}{\rho c}$ for alternating currents. Morse and others have shown that the relation fails in many cases, and cannot predict the results in any way.

ARTIFICIAL ABSORBER

The study of the acoustic impedance of different materials has shown us the dependence of the absorbing power of a material on its physical properties.

It suggests that if we can by some means control the physical properties of a material we can adjust its sound-absorbing power to any desired value. To achieve this ideal condition we thought of making an artificial absorber by collecting together a number of fine vaccine tubes of equal length and of almost same diameter. By varying the number of tubes we could vary the porosity and mass reactance. A brass plate of thickness 5 mms. and diameter 5·6 cms. was taken and equidistant holes of the requisite number were bored in the plate. Then fine capillary tubes were inserted in these holes; since the tubes did not fit the holes tightly and a thin annular space could remain between the tube and the hole, the chinks were closed by beeswax and resin at the back side of the plate, so that the front surface remained clean and smooth except for capillary spaces. In this way we prepared a number of plates having different number of tubes inserted in them. Since the plate differed only in the number of tubes fixed to them and were otherwise similar, we could thus imitate a material of known variable porosity and mass reactance.

In order to study the effect of changing the mass reactance and stiffness reactance separately, we took a thin plate only 2 mms. thick and inserted 55 tubes in it in the same manner.

For doing the experiment one of these systems was attached to a long hollow piston the back side of which was open to a long air column. Whatever sound happened to pass through the tubes, was never reflected back and was fully absorbed because there was no rigid backing at the ends of the tubes.

THEORY

To find the terminal impedance in this case let us first consider what happens when sound waves fall normally upon the perforated plate. Since the volume current at the junction of the standing wave tube and the capillary tube system should be the same on both sides of the surface of the plate A,

(1) the volume current towards the right due to incident plus the reflected wave must be equal to the volume current through the openings; also

(2) the acoustic wave pressure on both sides of the surface must be the same.

Let us assume plane progressive waves inside the standing wave tube, then

$$(\dot{\xi}_1+\dot{\xi}_r)\, a = n\pi r^2 \dot{\xi}_2,$$

where $\dot{\xi}_1$, $\dot{\xi}_r$ and $\dot{\xi}_2$ represent the particle velocities of the incident, reflected and transmitted waves respectively; a represents the cross-sectional area of the piston, r represents the average radius of the capillary tubes, and n is the number of tubes.

From the second condition we find that

$$p_1 + p_r = p_2$$

where p_1, p_r and p_2 represent the pressures due to the incident, reflected and the transmitted waves respectively.

Since we assume plane waves inside the standing wave tube, we can represent pressure by the product of impedance and volume current; *i.e.*, $p = Z_0\dot{\xi}$

therefore $$Z_0\,(\dot{\xi}_1 - \dot{\xi}_r) = \dot{X}Z_2 = \pi r^2 \dot{\xi}_2\, Z_2$$

Here Z_0 is equal to ρc, and Z_2 is the impedance of a *single opening*.

If Z'_2 represents the specific acoustic impedance of the opening, then

$$Z'_2 = Z_2 \pi r^2$$

So putting Z'_2 for $\pi r^2\, Z_2$ and solving the above equations,

we get $$\frac{\dot{\xi}_r}{\dot{\xi}_1} = \frac{A' - Z'_2/Z_0}{A' + Z'_2/Z_0}$$

where $$A' = \frac{n\pi r^2}{a} = \frac{\text{Porous area}}{\text{Total area}}$$

but $$\alpha e^{j\phi} = \frac{\dot{\xi}_r}{\dot{\xi}_1}$$

and therefore $$= \frac{A' - Z'_2/Z_0}{A' + Z'_2/Z_0} \quad \ldots \ldots \quad (2)$$

or $$\frac{Z'_2}{Z_0} = \frac{A'(1 - \alpha e^{j\phi})}{1 + \alpha e^{j\phi}} \quad \ldots \ldots \quad (3)$$

Equating the real and the imaginary parts on the two sides of equation (3), we find that the resistance and reactance terms of the specific acoustic impedance in the case of glass tubes are given by

$$\left.\begin{aligned} \frac{Z_r'}{Z_0} &= \frac{A'(1-\alpha^2)}{1+\alpha^2+2\alpha\cos\phi} \\ \frac{Z_x'}{Z_0} &= \frac{-2A'\alpha\sin\phi}{1+\alpha^2+2\alpha\cos\phi} \end{aligned}\right\} \quad \ldots \ldots \quad (4)$$

respectively.

After finding out the constants of the receiver, we determined the values of A and B at each frequency experimentally. Then knowing A and B, we calculated the values of α and ϕ for different frequencies with the help of the relation

$$\alpha\, e^{j\phi} = e^{-2(A + jB)}$$

Substituting these values of α and ϕ in equations (4) we evaluated the real and imaginary terms of the specific acoustic impedance of the plate and glass tubes system.

Experiment was done with all the plates in the manner described before at different frequencies and results calculated.

Table II gives the results for four plates of which three are thick plates, having 128, 88 and 55 tubes respectively inserted in them, and the fourth is a thin one having 55 tubes. Slight deviation of the results at places occurs because the order of resistance and reactance to be measured is very small. It is so small that throughout the calculation seven figure logarithms had to be used.

Length of the tubes was practically 10 cms.

Radius of the tube ... ... ·035 cm.

Radius of the piston ... 2·8 cm.

Table II

Number of tubes ... ... 128 ;
value of A' ... ... ·02

λ in cms.	α	Zx'/Z_0	Zr'/Z_0	ϕ
37	·982	−·935	·988	179° 0′
36	·980	−·872	·999	179° 0′
34	·979	·343	1·919	180° 13′
32	·981	−·884	1·062	179° 5′
31	·987	−·848	·268	177° 36′
30	·983	−·805	·32	177° 36′
28	·972	−·614	·381	177° 20′

Number of tubes 88;
value of A' ... ... ·01266.

λ in cms.	α	Zx'/Z_0	Zr'/Z_0	ϕ
37	·986	−·722	·599	179° 0′
36	·987	−·783	·575	179° 0′
34	·984	·331	1·546	180° 12′
32	·987	−·662	·623	179° 0′
31	·987	−·918	·583	178° 48′
30	·986	−·894	·583	178° 48′
28	·977	−·596	·515	178° 30′

Number of tubes ... ... 55 ;
value of Λ' ... ... ·008594.

λ in cms.	α	Zx'/Z_0	Zr'/Z_0	ϕ
37	·989	−·584	·357	179° 0′
36	·990	−·465	·126	178° 0′
34	·988	·412	1·417	180° 12′
32	·986	−·524	·304	178° 36′
31	·989	−·677	·362	178° 50′
30	·989	−·695	·362	178° 48′
28	·983	−·453	·266	178° 22′

Thin plate

Number of tubes ... ... 55 ;
value of Λ' ... ... ·008594.

λ in cms.	α	Zx'/Z_0	Zr'/Z_0	ϕ
37	·977	−·343	·428	178° 58′
36	·984	−·408	·192	178° 0′
34	·975	·108	·673	180° 14′
32	·976	−·217	·699	179° 34′
31	·977	−·392	·426	178° 48′
30	·983	−·527	·421	178° 48′
28	·965	−·009	·296	179° 21′

The values of resistance and reactance for different number of tubes at different wavelengths have been plotted in Fig. 2. Solid lines show this variation in the case of thick plates and thick dotted lines show it for thin plate having 55 tubes inserted in it.

Before discussing these results, let us consider what should happen according to the theory when the sound waves fall normally on the plate. After these theoretical considerations, we shall be able to compare the results with those predicted by theory.

Theoretical considerations.—From the theory of the resistance to alternating motion of air in such tubes, there are two relations which give specific acoustic impedance according as the tubes are narrow or of moderate width. The criterion which separates the two cases is the quantity $\left| r\left(\frac{j\omega}{\nu}\right)^{\frac{1}{2}} \right|$ in which r is the radius of the tube and ω is 2π times the frequency $(2\pi f)$. If this expression is less than unity, the tube according to Rayleigh[6] is effectively narrow (Case I) ;

if it lies between 1 and 10, it falls in the moderate width class (Case II). The specific acoustic impedances for the two cases are given by

$$\frac{Zr'}{Z_0}=\left(\frac{4\nu}{\omega r^2}\right)^{\frac{1}{2}}(1-j) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$\frac{Zr'}{Z_\theta}=1+(1-j)\left(\frac{\nu}{2\omega r^2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

respectively. If, however, the tubes are of finite length, we must replace these equations by others involving the impedance of a finite pipe. Then the impedance factor given above must be multiplied by

$$-j\cot\left[\left\{\frac{\omega}{rc}\left(\frac{4\nu}{\omega}\right)^{\frac{1}{2}}(1-j)\right\}l\right] \text{ in Case I,}$$

and $$-j\cot\left[\left\{\frac{1}{rc}\left(\frac{\omega\nu}{2}\right)^{\frac{1}{2}}(1-j)+\frac{\omega}{c}\right\}l\right] \text{ in Case II,}$$

to obtain the appropriate values of the relative impedances.

In our case there was no rigid backing at the end of the tubes ; so whatever energy once passed through the tubes was never reflected back and was fully absorbed. Thus the tubes acted as infinitely long tubes and hence equations (5) and (6) were applicable without the multiplying factor.

To find whether the tubes actually used in the experiment should be classed as narrow or of moderate width we evaluated the real part of $r\left(\frac{j\omega}{\nu}\right)^{\frac{1}{2}}$ after substituting the value of r, and ω when $f=1000$ cycles.

Radius r of the tubes $=\cdot035$ cm.

$$\nu=\cdot134\times4$$

$$\omega=2\pi f=2000\pi.$$

Therefore the real part of $r\left(\frac{j\omega}{\nu}\right)^{\frac{1}{2}}$

$$=\frac{1}{\sqrt{2}}r\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}$$

$$=2\cdot7.$$

This shows that the tubes are of moderate width and so relation (6) will give the specific acoustic impedances of these tubes. We calculated the acoustical impedances of the tubes at different frequencies from relation (6). Table III gives the values of resistance and reactance terms of the specific acoustic impedance according to this equation at different frequencies.

Table III

λ in cms.	Zr_2'/Z_0	Zx_2'/Z_0
37	1·193	−·193
36	1·190	−·190
34	1·185	−·185
32	1·179	−·179
30	1·174	−·174
28	1·168	−·168

The theoretical values given in Table III are plotted in Fig. 2 and are represented by the thin line in the figure. It will be seen from these curves, that

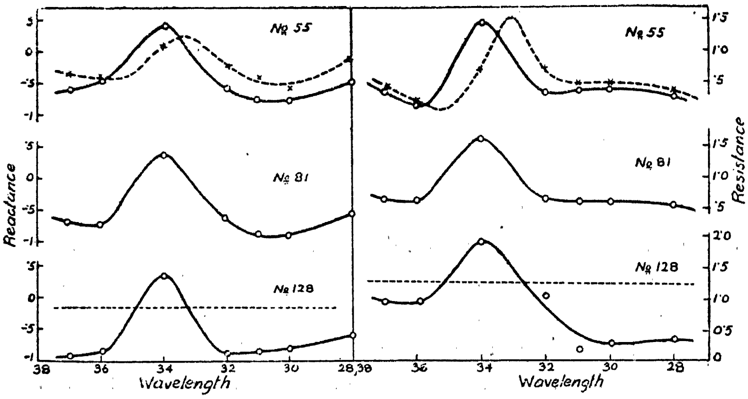

Fig. 2

The solid curves refer to thick plates and the dotted curves to thin plates. The numbers affixed to the curves denote the number of tubes used.

(1) the curves of Zr_2' and Zx_2' vary linearly with the wave-length, the former decreasing, and the latter increasing with the decrease of wave-length ;

(2) the values of Zr_2' and Zx_2' are independent of n the number of tubes;

(3) reactance always remains negative in the experimental range of wave-length.

The experimental curves on the other hand have pronounced maxima in the regions of $\lambda \sim 34$ cms., though their slopes in other regions have nearly the same values as predicted theoretically.

Also the experimental values of resistance in regions other than $\lambda \sim 34$ cms. are much lower than the theoretical values, and they go on diminishing as the number of tubes is reduced. The experimental values of resistance at the maxima also decrease as n decreases.

The experimental value of reactance near about $\lambda = 34$ cms. has a large positive value. In the other regions, however, the observed reactance is negative

as predicted theoretically though its values are numerically much greater than the theoretical values.

This divergence of the observed values in the neighbourhood of $\lambda=34$ cms., from the theoretical values suggests that at any rate in this wave-length region, the acoustic impedance found experimentally is not determined wholly by the absorption of sound in tubes as contemplated in the theory.

The brass plate to which the tubes are fixed is presumably not quite rigid and may therefore execute transverse vibrations which will naturally be communicated to the tubes. There will, in consequence, be an anti-resonance effect between the plate and the tubes and this will contribute appreciably to the observed impedance.

As we shall see in the next section the observed changes in ϕ and α also lend support to the above suggestion regarding the origin of the extra impedance. We postpone the consideration of this extra impedance to a later part of the paper.

Let us consider next the phase change ϕ and the reflection coefficient α. According to equation (2), they are given by

$$\alpha e^{j\phi}=\frac{A'-Z_2'/Z_0}{A'+Z_2'/Z_0}$$

Calculating ϕ and α for different wave-lengths and two different numbers of tubes, and from the values of Z_2'/Z^0 given in Table III it is found that for $n=128$,

at $\lambda=37$ cms., $\phi=179°42'$ and $\alpha=\cdot968$
at $\lambda=28$ cms., $\phi=179°44'$ and $\alpha=\cdot966$
For $n=55$, at $\lambda=37$ cms., $\phi=179°53'$ and $\alpha=\cdot988$.

This shows that according to the theory, ϕ and α both should increase as the number of tubes decreases. Experimentally we find that α increases as the number of tubes diminishes, but the changes in ϕ are not very appreciable.

Also according to the above calculation α should decrease and ϕ should increase when the wave-length decreases. Experimentally α does decrease with the decrease of wave-length but nothing much can be said about the changes in ϕ.

Table II shows that in the regions near about $\lambda=34$ cms., the value of ϕ shows a maxima and it reaches very close to 180°. Also the value of α has diminished considerably in this region. The occurrence of this maxima and the decrease of α at about $\lambda=34$ cms. point again to an extra contribution to the observed impedance over and above that due to the absorption of sound by the tubes.

We shall now consider the question of the extra-contribution to the observed impedance. As is mentioned already the theoretical expressions given in the earlier parts of this paper are based on the assumption that the plate to which the

tubes are attached is perfectly rigid. Though this assumption will be true generally we should expect large deviations when the incident frequency happens to coincide with the natural frequency of vibration of the plate. This is evidently what is happening when the incident wavelength is in the neighbourhood of 34 cms.

We shall now proceed to investigate the consequences of such vibrations being set up in the plate. Let M denote the solid area and N the porous area of the plate. Then the volume current due to the solid area will be $\dot{\xi}_m$ and that due to the porous area will be N $\dot{\xi}_2'$. Therefore the total volume current when the vibration of the plate is also considered, will be

$$M\dot{\xi}_m + N\dot{\xi}_2' \quad \ldots \quad (a)$$

Since the pressures at the solid area, porous surface and just inside the tubes are all equal, and further

$$\frac{p}{\dot{\xi}_m} = Z_m,$$

$$Z_2'\dot{\xi}_2' = \dot{\xi}_m Z_m,$$

and hence,

$$\dot{\xi}_m = \frac{Z_2' \dot{\xi}_2'}{Z_m}$$

Z_m being the mechanical impedance of the plate and the tubes combined. Putting this value of $\dot{\xi}_m$ in (a) we get,

$$\text{total volume current} = \frac{MZ_2'\dot{\xi}_2'}{Z_m} + N\dot{\xi}_2$$

Since the pressure remains the same, it can be put equal to $\xi_2' Z_2'$, therefore

$$Z' = \frac{p}{\text{Volume current}} = \frac{\dot{\xi}_2' Z_2'}{\dot{\xi}_2' \left(N + \frac{MZ_2'}{Z_m}\right)}$$

$$= \frac{Z_2'}{N + \frac{MZ_2'}{Z_m}}$$

$$= \frac{1}{\frac{N}{Z_2'} + \frac{M}{Z_m}}$$

or

$$\frac{1}{Z'} = \frac{N}{Z_2'} + \frac{M}{Z_m}$$

where Z' is the total terminal impedance. Hence, it shows that the mechanical impedance of the plate acts in parallel with the acoustic impedance of the tubes. Knowing Z' and Z_2', Z_m/Z_0 can be found out by the relation (7);

$$\frac{Z_m}{Z_0}=\frac{Z_2'Z'M}{(Z_2'-NZ')Z_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{7}$$

In this manner, the values of Z_m/Z_0 have been calculated for different wave-lengths for two values of n, namely, 128 and 55, using for the purpose the values of Z'/Z_0 from Table II and those of Z_2'/Z_0 from Table III. The calculated values are given in Tables IV and V :

Table IV.

Number of tubes $n=128$.
Value of $M=24{\cdot}14$ sq. cms.
Value of $N=\cdot49$,, ,,

λ	Z_{mr}/Z_0	Z_{mx}/Z_0
37	15·26	−50·16
36	18·8	−48·65
34	72·47	+109·5
32	91·74	−52·72
30	·42	−23·83
28	5·7	−19·99

Table V.

Number of tubes $n=55$.
Value of $M=24{\cdot}415$ sq. cms.,
Value of $N=\cdot2156$,, ,,

λ	Z_{mr}/Z_0	Z_{mx}/Z_0
37	7·83	−17·09
34	42·64	19·54
32	6·64	−14·34
30	7·21	−21·27
28	5·93	−12·24

By comparing the values in Table IV for $n=128$, with those for $n=55$ in Table V, particularly in the region $\lambda=34$ cms., it will be seen that the mechanical impedance of the system decreases with the decrease in the number of tubes. This is just what we should expect if the system consisting of brass plate and the tubes is set into vibration ; since by decreasing the number of tubes, the effective mass and the stiffness of the whole system decrease.

Having been led by the experimental data to the conclusion that the mechanical vibration of the system as a whole is responsible for the anomalous behaviour in the neighbourhood of $\lambda=34$ cms., we now proceed to discuss in detail the influence of these vibrations on the mechanical impedance of the combined system.

It can be seen from Tables IV and V that there is a sharp rise in the value of mechanical resistance in the region $\lambda=34$ cms. At $\lambda=30$ cms. it passes through a minimum and then increases slightly on further increase of frequency. The thick dotted curve shows this variation of resistance for $n=128$.

It is also seen that Z_{m_x} is negative and small in magnitude at low frequencies of about 800 cycles/sec. but attains a positive peak value at about 1000 cycles, then falls down and attains a negative peak value at 1062 cycles in the case of 128 tubes. At frequencies higher than 1062 cycles the magnitude of reactance decreases steadily and tends towards positive values.

This large variation of mechanical resistance and reactance cannot be due to the vibration of the plate alone ; since (1) the resistance due to the vibration of a plate is very small ; (2) the reactance due to the vibration of a plate is given by

$$Z_{p_x}=\left(m\omega-\frac{S}{\omega}\right)$$

which passes from negative to positive values when the frequency of the incident sound crosses the natural frequency of vibration of the plate. Once it becomes positive there are little chances of its again becoming negative within the given range of frequency. Experimentally it acquires a large negative value immediately after attaining a positive peak value.

These effects definitely show that the tubes also take part in the transverse vibration of the system.

Behaviour of the tubes when the plate vibrates.—When the plate vibrates, it buckles in and out and its each element is tilted. This tilt gives rise to a couple acting at the point of attachment of the glass tubes to the plate and causes transverse vibrations of the tubes as the ends of the tubes are free to move.

Let us represent an element of the plate by PQ, and the glass tube by a line OM perpendicular to PQ, the elasticity of the joint and the capillary tubing by a spring S_2, and the effective mass of the tube by m'.

Fig. 3(*a*) shows the steady state of the plate and the tube. When the element PQ is tilted and takes the position P′Q′, making an angle θ_1 with PQ, then OM will take the position OM′ if there is no bending of the spring, but because the spring bends through an angle θ_2, M′ takes up the position M″ and

OM″ makes an angle θ_3 with OH, the horizontal axis, and OM′ makes an angle θ_1 with OH.

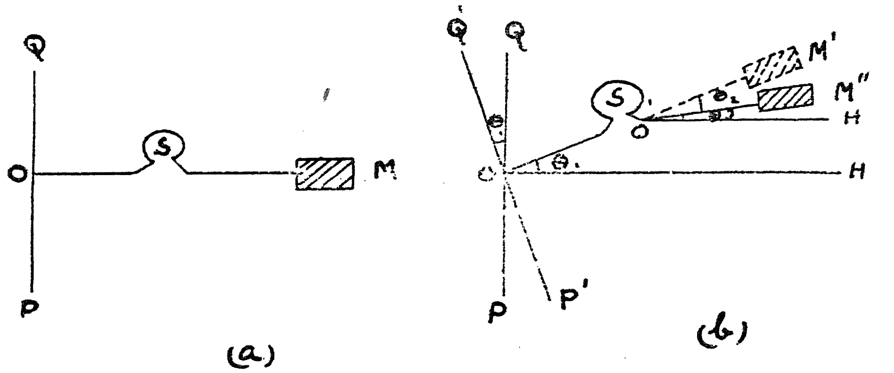

Fig. 3

Hence, $\dot{\theta}_1 = \dot{\theta}_2 + \dot{\theta}_3$ . . . . (a)

Further, $Z_{p_e} \dot{\theta}_1 = \frac{\Gamma}{l^2} - S_2 \dot{\theta}_2$ . . . . (b)

where Γ is the external couple on the plate. This couple is acting on each element of the plate and is due to the pressure variations caused by the sound waves on the surface of the plate. S_2 represents the spring constant and Z_{p_e} is the mechanical impedance of an element of the plate, equal to $\frac{Z_p}{n}$. Also the couples due to the bending of the spring and that due to the motion of the mass must be equal ;

hence, $S_2 \dot{\theta}_2 = Z_3 \dot{\theta}_3$ . . . . (c)

On eliminating $\dot{\theta}_2$ and $\dot{\theta}_3$ from equations (a), (b) and (c) we get

$$\dot{\theta}_1 = \frac{\Gamma / l^2}{Z_t + \dfrac{1}{\left(\dfrac{1}{Z_3} + \dfrac{1}{S_2}\right)}}$$

Thus we find that the spring acts as a shunt across the reactance of the tubes. Here Z_3 represents the mechanical impedance due to the transverse vibration of the tubes. Therefore the combined mechanical impedance Z_m is given by

$$Z_m = Z_p + Z_t$$

where Z_t is the mechanical impedance due to the transverse vibration of n number of tubes and is given by

$$Z_t = nZ_1$$

where
$$\frac{1}{Z_1} = \frac{1}{Z_3} + \frac{1}{S_2}$$

Since the impedance due to the transverse vibrations of the tubes will consist of a mass reactance $m'\omega$ and resistance r and the joint will act only as a capacity C_2, the above equation can be written as

$$\frac{1}{Z_1} = \frac{1}{r + jm'\omega} + jc_2\omega, \quad c_2 = \frac{1}{S_2}$$

$$= \frac{r + jm'\omega}{(1 + m'c_2\omega^2) + jrc_2}$$

or
$$Z_t = n\left\{\frac{r + j[m'\omega - c_2\omega(\omega^2 m'^2 + r^2)]}{\omega^2 r^2 c_2{}^2 + (m'c_2\omega^2 - 1)^2}\right\}$$

By putting suitable values for C_2, r and m' in the above equation, we can calculate the values of Z_{t_x} and Z_{t_r} for different frequencies. Hence, taking $C_2 = 10^{-5} \times 2{\cdot}5$, $r = 1$ ohm and $m = {\cdot}001$ gm., we calculated Z_{t_r}/Z_0 and Z_{t_x}/Z_0 for $n = 128$ at different frequencies. Table VI gives these calculated values.

Table VI

Frequency f	Resistance Z_{t_r}/Z_0	Reactance Z_{t_x}/Z_0
796	19·4	33·7
923	65·2	50·7
955	93·2	41·5
971	107·0	29·0
1000	121·0	−8·3
1007	120·0	−18·8
1114	37·0	−64·5
1274	7·5	−37·5

These values are plotted in Fig. 4. Solid lines in this figure show the curves for Z_{t_r} and Z_{t_x} and the dotted lines show the variation of Z_{m_r} and Z_{m_x} found experimentally with frequency as given in Table IV. It may be again mentioned here that the experimental values of Z_{m_r} and Z_{m_x} give the mechanical impedance of the plate and the tubes combined whereas Z_t is the theoretical value of tubes only,

i.e.,
$$Z_m = Z_t + Z_p$$

Comparing these theoretical and experimental curves of resistance, we find that the form of the two curves is similar but the experimental values of Z_{m_r} are

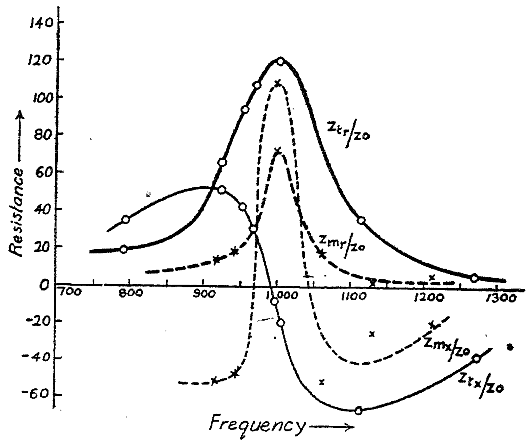

Fig. 4

smaller than the calculated value of Z_{t_r}. This decrease of value can only be due to one reason, that all the tubes in the plate may not be equally effective in having the same value of Zr_1. It has been found that only those tubes which are at the centre take part in the transverse vibrations. So if we take the effective value of n to be $\frac{2n}{3}$, the values of theoretical resistance Z_{t_r} will come out of the same order as found experimentally.

Comparing the curves for Z_{m_x} and Z_{t_x} we find that (1) the values of Z_{m_x} are negative at low frequencies whereas those of Z_{t_x} are positive ; (2) the peak value of Z_{m_x} is attained at a frequency which is 50 cycles higher than that for Z_{t_x}.

The negative values of Z_{m_x} at low frequencies are due to the presence of Z_{p_x} and the capacity due to the back air chamber. The reactance due to the vibration of the plate below 900 cycles which is the natural frequency of the loaded plate, is negative and appreciable. The reactances Z_{m_x} and Z_{t_x} attain positive peak values at different frequencies ; this is due to the presence of Z_{p_x}. Since Z_{m_x} is always given by the sum of Z_{p_x}, and Z_{t_x} and very near 1000 cycles where both Z_{t_x} and Z_{p_x} are positive, Z_{m_x} will have a large positive value. We find this to be the case at about 1000 cycles. At 1100 cycles Z_{m_x} attains a maximum negative value because at this frequency Z_{t_x} has its negative peak

value. At this frequency the negative value of Z_{m_x} is lower than the negative value of Z_{t_x}, since the positive value of Z_{p_x} has been added to Z_{m_x}.

It is also possible to explain the nature of the resistance and reactance curves for Z_{m_r} and Z_{m_x} by considering the effects of the Z_{t_r} and Z_{t_x} upon them. Thus by considering the transverse vibration of the tubes, we have been able to explain all the striking peculiarities in the observed values of mechanical impedance, α and ϕ in Table II and consequently the vibrations in the observed total acoustic impedance.

The ideas developed in the previous section regarding the influence of the vibration of the system consisting of the brass plate and the tubes on the mechanical impedance of the system receives much support from a comparison of the observed acoustic behaviour of the thin and thick plates.

Thin Plate

Comparing the results for thick and thin plates having 55 tubes, it is found from Table II that (1) the value of α decreases in the case of thin plate. The cause of this decrease is, that the amplitude of vibration of a thin plate is larger than that of a thick plate, so the transverse vibration of the tubes has got a much wider range of frequencies, and a larger fraction of energy is lost, hence α decreases.

Broken curves along the solid curves of resistance and reactance of $n=55$ (thick plate) in Fig. 2, show the resistance and reactance curves of thin plate for the same number of tubes.

Comparing the resistance curves we find that

(2) the resistance in the case of thin plate is more at any frequency than that for thick plate ;

(3) the frequency for maximum value of resistance is higher for thin plate than that for thick plate.

Comparing the reactance curves we find that

(4) the curve for thin plate is flatter than that for thick plate, since the values for thin plate have diminished numerically. The effect of diminishing the thickness of the plate on the reactance curve is same as of reducing the number of tubes.

(5) Also in the case of thin plate the frequency for maximum reactance is a little lower than the frequency for maximum resistance. This change of frequency we could not detect in the case of thick plate.

As far as point (2) is concerned we have already said above that the amplitude of vibration of the thin plate increases and so the loss of energy has increased

and thus the plate resistance Z_{p_r} becomes appreciable and is added to the mechanical resistance of the system. Hence Z_r' for thin plate increases.

We also know that when the thickness of the plate will decrease, the effective mass m and stiffness S both will decrease, so the numerical value of the reactance due to the vibration of the plate will decrease at any frequency. Consequently the total numerical value of Z'/Z_0 decreases, and the reactance curve becomes flatter in the case of thin plate as noted in point (4).

When the plate becomes thin the coupling constant C_2 should diminish. So to get the approximate resonance frequency

$$m' C_2 w^2 = 1$$

Therefore when C_2 diminishes, w must increase, *i.e.*, the frequency for maximum value of resistance Z_{t_r} must increase. This increase of frequency follows from the consideration of Z_{t_x} also. When Z_{t_x} becomes equal to zero,

$$m' \omega = C_2 \omega (r^2 + m'^2 w^2)$$

or

$$r^2 + m'^2 w^2 = \frac{m'}{C_2}$$

As C_2 diminishes and m' and r remain constant, w must increase when Z_{t_x} is equal to zero. This shows that the frequency at which Z_{t_x} becomes zero, must increase. Therefore it is possible that the frequency for maximum value of Z_{m_x}/Z_0 will shift in the case of thin plate.

Now let us see how the decrease of C_2 affects the values of Z_{t_r} and Z_{t_x}.

$$Z_{t_r} = \frac{rn}{\omega^2 C_2{}^2 r^2 + (m' C_2 \omega^2 - 1)^2}$$

In this equation for maximum value of Z_{t_r}, $m' C_2 w^2 = 1$, but $\omega^2 C_2{}^2 r^2$ will decrease when C_2 decreases since $\omega^2 = \frac{1}{m' C_2}$. So Z_{t_r} will increase a little. This increase also is partly responsible for increasing the total resistance Z_r'/Z_0 in the case of thin plate.

$$Z_{t_x} = \frac{[m' - C_2 (r^2 + m'^2 \omega^2)] \omega n}{\omega^2 C_2{}^2 r^2 + (m' C_2 \omega^2 - 1)^2}$$

The denominator diminishes, in the numerator the negative term diminishes but the positive term increases. This shows that the positive values of Z_{tx}/Z_0 before the resonance frequency of the tubes increase but the negative values after the resonance frequency decrease in the case of a thin plate. In other words the reactance curve for Z_{t_x} shifts towards the positive side of reactance when the plate becomes thin. We have already shown that frequency for $Z_{t_x}/Z_0 = 0$ shifts

slightly to higher frequency side, therefore all these facts show that the curve of Z_{t_x}/Z_0 for thin plate will be parallel to that for thick plate when n remains the same.

The frequency where Z_{m_x} becomes maximum is a little lower than the frequency at which Z_{m_r} has attained a maximum value as noted in point (5). This is exactly in accordance with the theory. We can see theoretically that for maximum Z_{m_x} and consequently for maximum Z_x' the frequency should be lower than that for maximum Z_{m_r} or Z_r' in every case. Due to some experimental errors we could not detect it in the case of thick plate, but we could get it in the case of thin plate.

Thus we find that all the points observed experimentally are explained when the transverse vibrations of the plate and the tubes and their combined effect are considered.

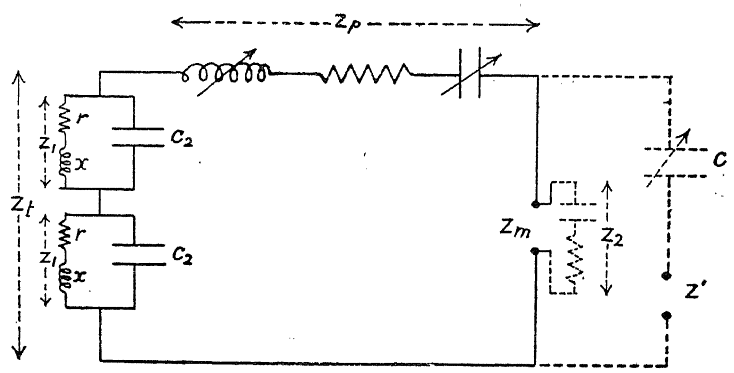

Fig. 5
Equivalent electrical diagram showing the acoustic impedance of the plate and tube system.

The behaviour of the plate and tube system can be represented by an equivalent electrical circuit by connecting a condenser and a resistance representing Z_2' in parallel with Z_m and then in series with the resultant is connected a variable condenser representing the back air chamber. Then whole of it represents Z'. Total value of Z_m is represented by Z_p connected in series with n number of Z_1, where Z_1 consists of a capacity due to the joint, connected in parrallel with the impedance due to the transverse vibrations of the tubes. The dotted circuit in Fig. 5 together with the circuit for Z_m shows completely the equivalent electrical circuit of the plate and tube system.

To test the validity of the above reasoning regarding the sudden increase of resistance and reactance at a certain frequency due to the transverse

vibrations of the whole system, we closed the ends of the tubes and embedded them in beeswax and resin. This made the whole system perfectly rigid and incapable of transverse vibration, and the closed ends of the capillary tubes act like a perfect reflector. The sensitivity of the method was increased by using a small amplifier in the detecter circuit. This enables us to do the experiment within a wider range of frequency, *i.e.*, between 500—1500 cycles. Beyond 500 cycles the change of inductance exceeds the inductometer scale and so readings cannot be taken. Experiment can be done even beyond 1500 cycles, but it is found that the mechanical reactance of the receiver passes from positive to negative values near about 1200 cycles. This seems to be the resonance frequency of the receiver and so near this frequency we cannot get the true value of impedance. For this reason we could only do the experiment up to 1200 cycles.

We repeated the experiment with this perfectly rigid system and found the values of resistance and reactance at different frequencies. It is found that the resistance increases from ·3 to ·4 ohms as the frequency varies from 500 to 1100 cycles. Beyond 1100 cycles it decreases suddenly as the frequency passes through the resonance frequency of the receiver.

The reactance is negative and practically of the same order but at times we get a positive value for reactance. This is due to the fact that the phase change is very near 180°, and it is not possible to determine the phase change very accurately. The changes are very small as the ratio of the porous area to the total area is only 0·01122. According to the formula (4) the value of reactance depends upon the accurate evaluation of the phase change. So it is not possible to determine such small changes of reactance accurately.

Theoretically we find that according to the Rayleigh's formula, the specific acoustic impedance of this system will be given by

$$\frac{Z'}{Z_0} = -j\left\{1+(1-j)\frac{1}{r}\sqrt{\frac{\nu}{2\omega}}\right\} \cot l\frac{\omega}{c}\left\{1+\frac{1}{r}\sqrt{\frac{\nu}{2\omega}}\left(1-j\right)\right\}$$

The real and imaginary terms of the right-hand side expression give us the values of resistance and reactance components of the acoustic impedance of this system. For this particular case $r = \cdot 042$ cms. and $l = 13$ cms. Putting these values in the above formula we find that the resistance varies from ·3 to ·47 ohms as the frequency changes from 600 to 1500 cycles. It is exactly what we get experimentally. The reactance, according to this calculation, varies from $-\cdot 46$ to $-\cdot 1$ as the frequency changes from 600 to 1500 cycles. This is also fairly in agreement with the experimental results.

This agreement of the theoretical and the experimental values in this case further strengthens our arguments regarding an anti-resonant effect taking place in the case of the first system in which the tubes are not rigidly fixed to the plate.

Interpretation of the acoustic impedance of the plate and tube systems on the basis of their physical properties.—To treat this case theoretically by finding its porosity flow resistance etc. as in the case of commercial materials, we find that the porosity is defined as the ratio of the volume of void to the volume of the whole material, or it is the same thing as,

$$P = \frac{\text{porous area}}{\text{total area}} = A'$$

since the length is same in both cases.

Flow resistance is the ratio of the pressure drop per unit thickness to the volume flow of air through the material in cubic cms. per sec. per cms.

i.e.,
$$r = \frac{\dfrac{dp}{dx}}{\dfrac{Q}{n\pi a^2}}$$

where dp/dx is the pressure drop and $\dfrac{Q}{n\pi a^2}$, the volume current per unit area. Volume current through a single opening is

$$Q_1 = \frac{dp}{dx} \frac{\pi a^4}{8\eta}$$

then
$$Q = Q_1 n$$

therefore
$$r = \frac{n\pi a^2 \times 8\eta}{\pi a^4 n} = \frac{8\eta}{a^2}$$

We find that porosity $P = A' = \cdot 02$ when $n = 128$, and flow resistance $\dfrac{r}{\rho C} = \cdot 03$.

Taking these values at $1 = 20$ cms. we find that for no value of m, the magnitude of the resistance or of the reactance comes out of the same order as given by experiment. Both are much higher than the experimental values. This shows that the formula given by Morse and others is for some reasons not applicable to very rigid surfaces like the plate and tubes system.

I take this opportunity to express my thanks to Dr. R. N. Ghosh for having suggested the problem and for taking interest in the work and to Prof. K. S. Krishnan for having taken the trouble of going through the manuscript.

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CHEMICAL EXAMINATION OF THE SEEDS OF *SOLANUM INDICUM* LINN. PART II. THE COMPONENT GLYCERIDES OF THE OIL AND A RE-EXAMINATION OF ITS ACIDS

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SUMMARY

[The fatty oil of the seeds of *Solanum Indicum* Linn. has been re-examined and found to contain lauric acid (0·27%), palmitic acid (4·98%), stearic acid (6·05%), arachidic acid (0 97%), oleic acid (35·13%) and linoleic acid (52·64%).

The glyceride structure of the oil has been determined by brominating the neutral oil and separating the liquid bromo-glycerides into simpler fractions and estimating their fatty acid composition. The component glycerides are as follows:—Trilinolin (1·6%), oleodilinolin (51·2%), dioleolinolin (10·4%), palmitodilinolin (3 4%), stereodilinolin (3·6%), arachidodilinolin (0·5%), palmito-oleolinolin (8·5%), stereoleolinolin (8·8%), arachido-oleolinolin (1·3%), palmitodiolein (4·9%), stereodiolein (5 1%) and archidodiolein (0·7%). The palmito-glycerides contain small amounts of lauric acid.]

In Part I, Saran[1] determined the component acids of the oil of *Solanum indicum*. About the same time*, a paper by Puntambakar and Krishna[2] on this subject also appeared. On comparing the data of the two investigations, it was found that whereas there was close agreement in the results so far as " liquid " acids were concerned, there were some marked differences in the nature of the "solid" acids. Saran found lignoceric acid, basing his conclusion on the equivalent weight determination and failed to detect palmitic acid. His conclusion regarding " lignoceric " acid, based on equivalent weight determination, was possibly justified in view of the fact that considerable uncertainty has been felt in the past as to whether the natural lignoceric acid is in reality the *n*-tetracosanoic acid or its branched-chain isomer with lower melting-point.

In view of these considerations we thought it necessary to repeat the work using a larger quantity of the oil. We now find that lignoceric acid does not in fact occur and the higher equivalent weight found in the earlier investigation was possibly due to some unsaponifiable matter having been carried over in the last fractions in the course of fractional distillation. The present results, however, agree fairly closely with those of Puntambakar and Krishna (Table X).

*Puntambakar and Krishna's paper appeared in the June (1941) issue of the Indian Chemical Society and Saran's in the August (1941) issue of the Proceedings of the National Academy of Sciences (India), but both the issues were received about the same time in September.

With one or two exceptions most of the work on fatty oils carried out in this country has consisted in the determination of their component fatty acids only. In England since 1927, Hilditch[3] *et al.* have introduced methods for the study of the glyceride structure of the fats and oils. The importance of this study becomes clear from the following consideration :—the fatty acids in oils and fats may be present either in the form of simple triglycerides (*e.g.*, triolein, tripalmitin etc.) or they may be distributed as heterogeneously as possible giving rise to a complicated mixture of mixed triglycerides. If the former rule were operative in nature, *i.e.*, each natural triglyceride molecule contained only one species of fatty acid, there would be no need for such a distinction as that pointed out above. Expressed on a molar percentage basis, the proportions of component fatty acids and of component glycerides would be the same. This, however, does not generally occur in nature and there is overwhelming tendency towards the production of mixed triglycerides in seed oils. In other words, the general "rule of even distribution" of fatty acids in the glycerides of the natural fats and oils has been shown by later research to be overwhelmingly operative in natural syntheses.

We have, therefore, also studied the structure of the component glycerides of the oil of *Solanum indicum*. The method is that employed by Vidyarthi and Mallya[4] to the determination of glyceride structure of the oil of Niger seed (*Guizotia abyssinica*), a semi-drying oil (I. V. 129·2) and consists in the fractional precipitation of brominated glycerides from unsaturated fats, using various solvents for separating the liquid bromoglycerides into simpler fractions.

Previously Suzuki[5] (*et al.*) and Hashi[6] brominated a number of vegetable and fish oils and determined their glyceride composition by the fractional crystallisation of the solid bromoglycerides from different solvents. They, however, could not examine the liquid bromoglycerides which resulted from the glycerides containing mono- and di-ethylenic acids. The present experiments as well as those of Vidyarthi and Mallya (*loc. cit.*) have, therefore, considerably advanced the the study of the glyceride structure of liquid fats.

The molar percentages of the fatty acids obtained by this method (Table XIII) agree closely with those obtained by the lead salt separation and ester fractionation methods (Table X). This shows the validity of the application of this newer method to the determination of glyceride structure of the oil.

The neutral oil was dissolved in 6 times its weight of dry acetone and chilled for 2 to 3 days at 0°C. Fully saturated and di-saturated glycerides were found to be absent as nothing precipitated out. The absence of fully saturated glycerides was further established by thrice oxidising the neutral oil with powdered potassium permanganate in acetone solution according to the method of Hilditch and Lea.[7] The neutral oil was brominated in petroleum ether at 0°C. The solid

and liquid bromoglycerides were resolved into a number of fractions with alcohol, alcohol and acetone mixture (1: 1) and acetone. All the fractions were debrominated, the component fatty acids determined and the amount of various glycerides in the fractions was calculated.

Solanum indicum oil was found to consist of the following glycerides :— Trilinolin (1·6%), oleodilinolin (51·2%), dioleolinolin (10·4%), palmitodilinolin (3·4%), stereodilinolin (3·6%), arachidodilinolin (0·5%), palmito-oleolinolin (8·5%), stereo-oleolinolin (8·8%), arachido-oleolinolin (1·3%), palmitodiolein (4·9%), stereodiolein (5·1%) and arachidodiolein (0·7%). The palmitoglycerides will contain small amounts of lauric acid.

The optical activity of the component glycerides of the oil.—If the glyceride molecule contained only one species of acid, its molecule will be symmetrical and it would, therefore, not exhibit any optical activity. But as we have seen that the 'rule of even distribution' of the fatty acids is operative in natural glycerides, optical activity may arise if the α and α' carbon atoms of the glycerol molecule are associated with two different acid radicals. In the present case of the monosaturated diunsaturated glycerides, the monosaturated oleolinolins amounting to 18·6 per cent.(mol.) of the oil fulfil this condition as the three acid radicals are different and, therefore, they should be optically active. In the case of the others both in this group and in the triunsaturated ones, optical activity will be present if α and α' carbon atoms carry two different acid groups, and it will be absent if they carry similar groups. We have, however, at present, no means of distinguishing between these two structures. The oil of *Solanum indicum* gives $+0{\cdot}1^{\circ}$ rotation for sodium light in a 1-decimeter tube. The actual notation must, however, be greater than this value as the oil contains a lævo-rotatory sterol, $[\alpha]_D^{38^{\circ}} = -11{\cdot}1^{\circ}$ (in chloroform), which must have partly neutralised the dextro-rotation of the oil. In any case it is, however, clear that these glycerides have very low rotatory power.

Experimental

The seeds* of *Solanum indicum* (Bijnore District, United Provinces) were obtained from the Punjab Ayurvedic Pharmacy, Amritsar, and identified by Mr. M. B. Raizada of Forest Research Institute, Dehra Dun. They were crushed and exhaustively extracted with petroleum ether (40—60°C). On distilling off the solvent, a yellow-coloured oil was obtained in 11% yield. It was purified by animal charcoal and Fuller's earth as a transparent light yellow oil which on

*The seeds in the previous investigation were obtained from the Punjab Ayurvedic Pharmacy, Amritsar, and were collected in the Kangra District.

keeping deposited a white solid (1%). The oil was semi-drying having the following physical and chemical constants :—

TABLE I

	I Bijnore sample (present work)	II Kangra sample	III Puntambakar & Krishna[2]
Sp. Gr.	at 18/18°C, 0·9125	at 34/34°C, 0·9159	at 15·5°C, 0·9156
Ref. Index.	at 19·5°C, 1·4668	at 33°C, 1·4652	at 15·5°C, 1·4671
Rotation	$[\alpha]_d^{25^\circ}$ +0·1°	...	+ 0·5°
Acid value	7·36	3·24	17·8
Saponification value	187·0	190·2	177·6
Acetyl value	3·2	...	44·4
Iodine value (Hanus)	125·0	136·25	121·5
Hehner number	95·55	94·1	...
Thiocyanogen value (24 hrs.)	77·02	...	...
Unsaponifiable matter from oil	1·0%	...	2%
Unsaponifiable matter from soaps	0·1%	0·09%	

1200 gms. of the oil were saponified with alcoholic sodium hydroxide and the non-saponifiable matter extracted with ether. The soap solution was then decomposed with dilute sulphuric acid when the fatty acid (1120 gms.) having the following constants were obtained (Table II).

TABLE II

	Bijnore sample	Kangra sample	P. & K.
Consistency	Liquid	Liquid	...
Neutralisation value	195·5	195·4	...
Saponification value	195·4	...	...
Sap. Equivalent	286·9	287·2	293·1
Iodine value (Hanus)	127·7	138·1	123·0
Thiocyanogen value (24 hrs.)	79·21	..	...

As the neutralisation value and the saponification value are the same, the fatty acids do not exist as anhydrides or lactones.

The mixture of acids was then separated into "solid" and "liquid" acids by the Twitchell's[8] Lead salt-alcohol process, the results of which are given in Table III :

TABLE III

Acids	Percentage	I. V.	N. V.	S. E.
Solid	12·27	3·56	194·7	283·1
Liquid	87·73	141·6	199·8	280·8

Examination of "Liquid" Acids

On oxidation with alkaline potassium permanganate,[9] a dihydroxy stearic acid (M.P. 131°C) and a tetrahydroxy stearic acid (M.P. 172°C) were obtained showing the presence of oleic and linoleic acids only.

The quantitative determination of these acids was done by the method of Eibner and Muggenthalor,[10] modified by Jamieson and Boughmann,[11] the results of which are given below (Table IV) :—

TABLE IV

Wt. of acids taken	5·7560 gms.
Wt. of tetrabromide	4·8937 gms.
M.P. of the tetrabromide	114°C
Wt. of the residue (di+tetrabromides)	6·1158 gms.
Percentage of bromine in the residue	43·20
Wt. of tetrabromide in the residue	2·5030 gms.
Wt. of dibromide in the residue	3·6128 gms.
Wt. of oleic acid in the residue	2·305 gms.
Wt. (total) of linoleic acid	3·451 gms.

The following table gives the percentage of linoleic and oleic acids in liquid acids, in mixed acids and in the oil:

TABLE V

Acids.	Percentage in liq. acids.	Percentage in mixed acids.	Percentage in oil.
Linoleic	59·89	52·64	50·25
Oleic	40·11	35·13	33·57

The percentages of oleic and linoleic acids in the "mixed" acids were also calculated from the thiocyanogen and iodine values of the mixed acids (Table II) and show a very close agreement (Table VI) by the two methods.

TABLE VI

Acids.	Bromine addition method.	Calculation from SCN & I. V. of mixed acids.
Linoleic	52·64	53·52
Oleic	35·13	34·16

Examination of "Solid" Acids

The methyl esters (123·5 gms.) prepared from the "solid" acids (120 gms.) were fractionally distilled (10 mm. pressure). The results are given in Table VII.

TABLE VII.

Fraction No.	Boiling range.	Grams.
1	Upto 180°C	11·34
2	180—183°C	6·05
3	183—184°C	16·41
4	184—186°C	5·62
5	186—190°C	15·38
6	190—194°C	33·83
7	195—200°C	13·38
8	200—205°C	8·60
9	205—210°C	4·82
10	Residue: Undecomposed	4·37
	Residue: Decomposed	3·00
	Loss during distillation	0·70
		123·5

The saponification value, the mean molecular weight and the iodine value of all the fractions were determined and the amounts of various acids in the fractions were calculated according to the method of Jamieson and Boughmann[12] and reproduced in Table VIII.

TABLE VIII

Fraction No.	I. V.	S. V.	S. E.	Lauric acid		Palmitic acid		Stearic acid		Arachidic acid		Unsaturated acids	
				%	gms.	%	gms.	%	gms.	%	gms	%	gms.
1	1 1	218·2	257·0	21·87	2·479	71·85	8·147	...	...	...	...	0·78	0 089
2	1 6	206·6	271·5	...	...	88·98	5·384	4 71	0·285	...	...	1·13	0 068
3	1·9	207·6	270 3	...	...	93·46	15·330	...	...	...	...	1·34	0 220
4	2·2	197·1	284·7	...	...	44·96	2·527	48·55	2 729	...	...	1·55	0·087
5	2 6	197·3	284 3	...	...	45·82	7·047	47·40	7·288	...	...	1·84	0·282
6	2·9	191·0	293·7	...	...	14·26	4·822	78·91	26·690	...	...	2 05	0·693
7	3 0	191·2	293·5	...	...	14·57	1·950	78·54	10·510	...	...	2 12	0 284
8	3·4	184·9	303 4	...	...	...	...	73·74	6·342	19·33	1 663	2·40	0·207
9	3·9	176·0	318·7	...	...	...	...	21·15	1·020	71·69	3·455	2 75	0 133
10	16 7	...	...	...	...	...	...	...	...	...	3·680	...	...
					2 479		45 207		54 864		8·798		2·063

Total weight of saturated acids = 111 35 gms.

The esters from the different fractions were saponified and the liberated acids were fractionally crystallised from dilute acetone.

Fraction No. 1.—The mixture of acids melted at 56-57°C. It was repeatedly crystallised from dilute acetone and two fractions were obtained, one melting at 61°C and the other at 43°C. The higher melting fraction was palmitic acid (mixed M. P. with pure palmitic acid 61·5°C). The lower melting fraction was lauric acid (melting when mixed with pure lauric acid at 43·5°C).

Fraction No. 2.—The mixture of acids melted at 55°C. The acid on repeated crystallisation from dilute acetone proved to be palmitic acid (61°C); the melting-point was not depressed by the addition of palmitic acid.

Fraction No. 6.—The crude acids melted at 56°C. On repeated crystallisation from acetone, two products, one melting at 66°C and the other at 60°C, were obtained and were shown to be stearic and palmitic acids respectively by mixed melting-points.

Fraction No. 9.—The crude acids melted at 57°C. These were fractionally crystallised from acetone when two products were obtained, one melting at 75°C and the other at 66°C. The former seems to be arachidic acid but this could not be confirmed for lack of a pure sample of arachidic acid. The lower melting acid was stearic as confirmed by mixed melting-point.

Fraction No. 10. (Residue)—The acids were dark brown in colour and were extracted with petroleum ether (40—60°C) when a white mass was obtained which on repeated crystallisation from acetone melted at 74-75°C showing the presence of arachidic acid.

The percentages of various solid acids in mixed solid acids, in mixed acids and in the oil are given below (Table IX):—

TABLE IX

Acid.	Percentage in solid acids.	Percentage in mixed acids.	Percentage in oil.
Lauric	2·23	0·27	0·26
Palmitic	40·60	4·98	4·76
Stearic	49·27	6·05	5·78
Arachidic	7·90	0·97	0·93

The weights and molecular percentages of the constituent acids in mixed acids are given in Table X; the results of Puntambakar and Krishna *(loc. cit.)* being indicated in brackets.

TABLE X

Acid.	Wts. % in mixed acids.	Mol. % in mixed acids.
Linoleic	52·64 (49·5)	52·54
Oleic	35·13 (35·0)	34·82
Lauric	0·27 (0·6)	0·38
Palmitic	4·98 (7·2)	5·44
Stearic	6·05 (6·6)	5·95
Arachidic	0·97 (1·1)	0·87

The Unsaponifiable Matter.

The unsaponifiable matter from the oil consisted of two products. One of them, which was obtained by allowing the oil to settle melted at 216°C. The other product was obtained by extracting with ether the soaps from the oil. It melted at 145°C. and had $[\alpha]_D^{38} = -11{\cdot}1°$ in chloroform solution. The two products are phytosterols from their general colour reactions and will be later described in a separate communication.

The Component Glycerides.

The free acids in the oil were neutralised by treating with sodium carbonate and purifying the oil by animal charcoal and Füller's earth.

The neutral oil (100·2 gms.) was dissolved in 600 cc. dry acetone and chilled at 0°C in frigidaire for 2 to 3 days. A very minute quantity (0·08 gms.) settled down which was found to be unsaponifiable matter M. P. 215-216°C. Hence the oil does not contain appreciable quantity of any fully saturated or di-saturated glycerides.

100 gms. of neutral oil were dissolved in 1000 cc. dry acetone and oxidised by finely powdered potassium permanganate according to the method of Hilditch and Lea *(loc. cit.)*. The process was repeated three times when in the end 0·1 gm. of a neutral substance was obtained which was found to be non-saponifiable matter. This further confirmed the absence of fully saturated glycerides.

The neutral oil (100·1 gms.) was taken in a 2-litre flat-bottom flask and dissolved in 1000 cc. of petroleum ether (40—60°C). It was cooled down to −5°C and bromine was slowly added till the whole solution was permanently brown in colour. The mixture was left overnight at 0°C. in the frigidaire, the precipitate was filtered off and washed with chilled petroleum ether. It was treated with absolute alcohol when two fractions, one soluble in alcohol and the other insoluble were obtained. The petroleum ether filtrate (and the washings) was washed with sodium thiosulphate solution in order to remove the excess of bromine and on evaporating the solvent a dark coloured thick liquid was obtained. It was successively extracted with alcohol, alcohol+acetone mixture (1 : 1) and

acetone when 3 fractions were obtained. The scheme of separation is shown below :

NEUTRAL OIL (100·1 gms.)
Brominated in petroleum ether

Insoluble
treated with alcohol

Insoluble F_1
29·95 g.
dirty white
semi-solid

Soluble F_2
12·62 g.
Brown viscous
liquid

Soluble
extracted with
alcohol

Insoluble
extracted with
alcohol + acetone
(1 : 1)

Soluble F_3
17·47 g. Dark
brown viscous
liquid

Insoluble
extracted with
acetone

Soluble F_5 37·0 g.
viscous liquid

Soluble F_4 73·2 g.
very thick viscous
liquid

All these fractions were debrominated by taking them in methyl alcohol, adding zinc dust, saturating the solution with dry hydrogen chloride gas and heating for several hours under reflux on the water-bath. The debrominated products were saponified, the non-saponifiable matter removed and the acids liberated. The quantity of individual acids in each fraction was estimated by determining their saponification equivalents, iodine values and thiocyanogen values. The quantity of saturated acids being too small in these fractions for estimating them separately, they were considered as one and their mean molecular weight was determined on extracting with petroleum ether the oxidation products of each fraction with alkaline potassium permanganate according to the method of Lapworth and Mottram.[13] The analytical results are given in Table XI.

TABLE XI

	F_1	F_2	F_3	F_4	F_5
Wts. in gms.	29·95	12·62	17·47	73·2	37·0
Wts. of debrominated (glycerides with non-saponifiable matter) ...	18·9	7·5	13·0	39·8	20·9
Wts. of non-saponifiable matter ...	0·0001	trace	0·0012	0·088	trace
Wt. % of glycerides (free from non-saponifiable matter)	18·9	7·5	13·0	39·7	20·9
S. E. of liberated acids	275·1	281·2	285·9	282·1	281·8
I. V. of liberated acids	103·9	120·2	62·8	148·0	135·4
Thiocyanogen value of liberated acids	66·1	60·1	57·9	88·8	90·4
S. E. of saturated acids	259·8	281·0	288·7	...	...

TABLE XII

Mol. per cent. of acids in each fraction.

	F_1 19·3 % (mol.)	F_2 7·5 % (mol.)	F_3 12·8 % (mol.)	F_4 39·6 % (mol.)	F_5 20·8 % (mol.)
Linoleic	40·9	66·5	5·5	66·66	49·8
Oleic	30·7	...	59·4	33·33	50·2
Saturated	28·4	33·5	35·1	...	...

TABLE XIII

Mol. per cent. of acids on total acids.

	F_1	F_2	F_3	F_4	F_5	Mean.
Linoleic	7·9	5·0	0·7	26·4	10·4	50·4
Oleic	5·9	...	7·6	13·2	10·4	37·1
Saturated	5·5	2·5	4·5	...	...	12·5

TABLE XIV

Glycerides in	F_1 19·3	F_2 7·5	F_3 12·8	F_4 39·6	F_5 20·8	Mean. 100·0
(1) Fully saturated glycerides	nil	nil	nil	nil	nil	nil
(2) Di-saturated glycerides	nil	nil	nil	nil	nil	nil
(3) Monosaturated di-unsaturated glycerides						
(*a*) monosaturated dilinolin	...	7·5	...	...	...	7·5
(*b*) monosaturated oleo-linolin	16·5	...	2·1	...	...	18·6
(*c*) monosaturated diolein	...	...	10·7	...	...	10·7
(4) Triunsaturated glycerides						
(*a*) trilinolin	1·6	...	...	...	...	1·6
(*b*) oleodilinolin ...	1·2	...	...	39·6	10·4	51·2
(*c*) dioleolinolin ...	...	...	...	...	10·4	10·4

(1) By oxidation of the neutral oil with potassium permanganate in acetone.

(2) By chilling the neutral oil in acetone at 0°C.

(3 and 4) By calculating from the component fatty acids of the brominated glycerides in the oil.

All the saturated acids have been considered as one acid in the calculation of the monosaturated-di-unsaturated glycerides. The saturated acids are combined in the glycerides of the *Solanum indicum* seed oil as mono-saturated-di-unsaturated glycerides. It may be assumed that they are proportionately divided in monosaturated-dilinolin, monosaturated-oleolinolin and monosaturated-diolein glycerides. From the molecular percentage of the saturated acids in the oil, the component glycerides may be given as follows :—trilinolin (1·6%), oleodilinolin (51·2%), dioleolinolin (10·4%), palmitodilinolin (3·4%), stereodilinolin (3·6%), arachidodilinolin (0·5%), palmito-oleolinolin (8·5%), stereo-oleolinolin (8·8%), arachido-oleolinolin (1·3%), palmitodiolein (4·9%), stereodiolein (5·1%), arachido-diolein (0·7%). The palmito-glycerides will contain small amounts of lauric acid as in the above calculations the amount of lauric acid, being small, has been included in the palmitic acid.

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CYTOPLASMIC INCLUSIONS IN THE OOGENESIS OF CERTAIN *HYMENOPTERA*

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SUMMARY

The cytoplasmic inclusions, the processes of vitellogenesis, and the origin and role of the ndary nuclei have been investigated in the oocytes of three species of Hymenoptera, belonging ıree different genera. The secondary nuclei arise from nucleolar particles thrown out by the cipal nucleus, and they, in their turn, extrude fresh quantities of nucleolar substance in the asm. The nucleolar particles ultimately transform into proteid yolk spherules. In Scolia a nd generation of secondary nuclei arises from the nucleolar extrusions of the secondary nuclei he first generation. The Golgi bodies and mitochondria are not involved in the production of eid yolk, and there are no fat bodies.

Introduction

Special importance attaches to the study of the Hymenopterous oogenesis account of the presence of those enigmatic bodies called the secondary nuclei, although discovered as early as 1884, their mode of origin and functional ıificance remain obscure to this day. Besides, the eggs of Hymenoptera have n rarely investigated with the help of modern cytological techniques, and study of the cytoplasmic inclusions and their relationship with Blochmann's ebenkerne" promised to be fruitful.

The material selected for this study consists of three representatives of menoptera, namely, Vespa orientalis, Polistes hebraeus, and Scolia quadripustu- ıs. Da Fano, Cajal, Aoyama, Ludford's modification of Mann-Kopsch, and Kol- ıev techniques were employed for the demonstration of the Golgi bodies. The er-impregnated material was sometimes toned with gold chloride followed hypo and subsequently stained with safranin and light green or iron n haematoxylin. Similarly the osmicated material was bleached, whenever essary, with potassium permanganate and subsequently washed with oxalic l. Flemming without acetic acid, Altmann, and Champy were the chief methods l for the study of mitochondria. The sections were stained either with acid ısin differentiated with methyl green, aurantia or picric acid, or with iron

alum haematoxylin. Regaud, Regaud-Tupa, and Zenker-Helly were also employed, but preparations made according to these methods were used chiefly as controls. For the study of the general structure of the ovariole and the nucleolar extrusions Bouin's fluid was used as the fixative. Sections were stained with Mann's methyl blue eosin and Delafield's haematoxylin and eosin.

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Historical Account

The older works on insect ovaries are naturally confined to a consideration of histological details and, as such, are not of much importance to the present-day workers, except for the purposes of a historical survey. Some of these works, however, possess special features of interest, which cannot be ignored in a consideration of the general structure of the egg, and will be briefly mentioned along with the more recent works. Since this work is devoted to the oogenesis of some Hymenopterous insects, only works dealing with the oogenesis of this group of insects will be reviewed here.

Between 1884 and 1892 Blochmann published a series of papers devoted to an account of the morphology of insect eggs in which he drew attention to two remarkable structures, "Nebenkerne", *i.e.*, the secondary nuclei, and rod-like bodies, *i.e.*, the symbiotic bacteria, present in great abundance in the growing oocytes. According to him (1884), the nuclear membrane of the oocytes of a number of insects shows knotted thickenings, from which proceed buds, which gradually become bigger in size, detach themselves, and assume a nuclear structure. Blochmann's later account of the origin of the accessory nuclei (1889) is somewhat different from this.

Korschelt (1889) mentioned the presence of secondary nuclei in the eggs of Bombus and came to the conclusion that they were derived from the follicular epithelial cells.

Marshall (1907) described their origin and fate in Polistes pallipes and found that the structures thickly surrounding the egg nucleus are very much like true nuclei, possessing bodies like nucleoli and a nuclear network, but he could not definitely decide their nature and origin.

Marie Loyez (1908) gave a thorough account of the secondary nuclei in a paper devoted almost exclusively to a consideration of these structures. She definitely contradicted the idea of their nuclear nature and the suggestion that they arose from the germinal vesicle by budding, or that they were in-wandering follicular epithelial cells. She believed that they represented substances of fluid or granular nature which were secreted by the follicular epithelium, the nurse-cells and the oocyte nucleus and inflowed into the ooplasm. This secreted material

contains some chromatin, and is artificially transformed into a nuclear framework under the influence of the fixatives.

She also observed chromatin extrusion at the beginning of the egg growth and found the ooplasm laden with grains staining like the chromatin grains of the nucleus. These, later, become surrounded by vacuoles.

Yolk bodies begin to arise before their degeneration, but all transitional forms between the degenerating secondary nuclei and yolk bodies are detected—a part of the yolk content of egg is furnished by them.

Buchner's (1912) paper giving an account of the symbiotic bacteria present in the eggs of a large number of insects, also deals with the secondary nuclei. Buchner was convinced that they were not follicular epithelial nuclei that had wandered into the egg, although they represented true nuclei.

Govaerts (1913) studied the oogenesis of Carabus, Cicindella, and Trichiosoma. He described mitochondria, nucleolar extrusions and secondary nuclei in Trichiosoma.

Marie Loyez (1913) in her account of the ovary of the queen of ant (Lasius niger) again mentioned the presence of the secondary nuclei. About their origin she writes that they arise from the chromatic substance extruded by the germinal vesicle in close contact with the latter long before the oöcyte is surrounded by a follicular epithelium.

Hegner (1915) described the differentiation of the various cellular elements of the ovarioles of some Hymenopterous eggs, and also studied the behaviour of the symbiotic bacteria infesting the eggs, and the accessory nuclei.

He showed that both the oocytes and the nurse-cells of the honey bee arise from the oogonia and were bound up in rosettes in the early stages. In the eggs of Camponotus he showed that the secondary nuclei arose early near the germinal vesicle. He considered their origin from the nuclear budding or by the intrusion of follicular nuclei improbable, and thought that they originated from the material emitted by the egg-nucleus into the cytoplasm.

In Apanteles glomeratus he described a similar nuclear behaviour, and mentioned the appearance of secondary nuclei in the anterior half of almost fully-grown oocytes. Their origin and fate was not determined. In Apanteles too he described the germ-cell determinant at the posterior end of the egg and showed that it underwent a series of complex changes.

In young oocytes of the mealy rose gally-fly, he showed that the secondary nuclei appeared to arise near the periphery from granules which stained like chromatin and which might have been extruded by the oocyte nucleus, the follicle cells, or the nurse-cells.

Buchner (1918) gave an exhaustive account of the accessory nuclei of the Hymenopterous eggs, based on his investigation of their origin and fate in a

number of species. Buchner came to the conclusion that the accessory nuclei represent true nuclei even though they are lacking in chromosomes. They are not produced by a direct budding of the germinal vesicle, but arise from chromatin granules lying at first free in the ooplasm. They arise from different sources in different species.

In Solenius, they are formed by the basi-chromatin granules which are originally produced in the nurse-cells and later wander forth into the ooplasm. He figures and describes secondary nuclei also in the nurse-cells of this species, and thinks that they arise either directly from the remarkable nucleoli which develop at the same time in the nurse-cell nuclei, or from the nurse-cell secretion.

In Andrena sp. the formation of the secondary nuclei occurs at the posterior pole of the egg, as well as around the oocyte nucleus. Their degeneration occurs through "Hyperchromasie," and some of them directly transform into yolk-bodies. The nurse-cells also contain similar bodies. The accessory nuclei of the posterior pole or the sides arise from chromatin granules either secreted in the first instance in the nurse-cell or produced locally in the ooplasm. In Osmia rufa they are produced entirely through the activity of the oocyte nucleus. In Sphecodes gibbus L. and Prosopis sinuata Schenk they arise from those secreted at first in the nurse-cells, and then discharged into the ooplasm. In Camponotus a participation of the nucleus in their formation seems probable.

A second generation is produced by the safranophil granules elaborated by the nurse-cells in a thick perinuclear plasma zone. Some of these bodies produce accessory nuclei even in this zone, and the nucleoli of the nurse-cells transform into similar structures by gradual swelling.

In Myrmecina latreillei Curt they arise from the chromatic droplets produced by the oocyte nucleus, and a second generation is produced later at the egg-surface by the nurse-cell secretion.

In Rhyssa the secondary nuclei arise from ooplasmic chromatin particles, probably of nuclear origin, or from nurse-cell chromatin secretion, or by the activity of oocyte nucleus. They ultimately transform into yolk-bodies. In Trogus the course of events is slightly different. In Tenthredo mesomelas they arise from the nurse-cell chromatic secretion.

In Arge pagana Panz the accessory nuclei arise at the egg-surface from small granules which originate there probably under the influence of material filtering through the follicle cells. These granules either form the nucleoli around which appear vacuoles and a nuclear framework, or transform directly into a nuclear structure by forming vacuoles inside.

Gatenby's contribution (1920) to the cytology of the egg of Apanteles deals especially with the origin and behaviour of the germ-cell determinant.

He finds that the germ-cell determinant contains no chromatin, no fat, no glycogen, and is revealed by methods which do not bring out mitochondria. Albuminous yolk bodies arise in the ground cytoplasm at the periphery, and are not derived from the mitochondria. He infers, however, that the secondary nuclei, lying at the periphery of the egg, may influence their formation. The secondary nuclei arise from minute chromatinic granules of nuclear origin. Later on, they degenerate.

Hogben (1920) showed that in Cynips all the three cellular elements of the ovariole arise from germ-cells, and found some granular bodies in the cytoplasm of growing eggs which stained like chromatin and were considered to be of nuclear origin. At the posterior end of the grown-up eggs of Rhodites he detected a region of deeply staining granules corresponding to the "germ-cell determinants." In Synergus rheinhardii also he described a body corresponding to this. In Synergus he detected the presence of the secondary nuclei, and stated that they arise from the chromatin-like particles ejected by the oocyte nucleus. Such granules occur also in the cytoplasm of egg of Orthopelma luteolator, but no secondary nuclei.

The secondary nuclei were described also in the eggs of Formica rufa. Regarding the subsequent fate of these bodies he, on the whole, was inclined to accept Loyez's views, that is, they are ultimately transformed into deutoplasmic spheres.

Mukerji (1930) using Feulgen's method for the detection of chromatin reported that the contents of the secondary nuclei of Apanteles did not give the characteristic chromatin reaction. It was also found that the germ-cell determinant was devoid of chromatin material.

Gresson (1930) used Feulgen's method on the eggs of Thrinax mixta and Allantus pallipes, and reported the absence of secondary nuclei from the cytoplasm of the nurse-cells and oocyte, and withdrew the former conclusions of Peacock and Gresson (1928) about the origin of the accessory nuclei from the oocyte, nurse-cells, and follicular cells.

Peacock and Gresson (1928) showed that in Tenthredinidæ the accessory nuclei have a three-fold origin, namely, from the nuclei of (1) nurse-cells, (2) oocytes, and (3) the follicle cells.

In older cells the cell-boundaries of the nurse-cells become indistinct and some of the cytoplasm together with the accessory nuclei pass down into the oocyte. In the final stages, the entire nutritive chamber becomes a mass of nucleated cytoplasm which passes into the oocyte. In the oocyte nucleus nucleolar budding and subsequent extrusion of the fragments were described.

Gresson's account (1929) of the nucleolar phenomena in the eggs of Thrinax macula and mixta and Allantus shows that in these forms the nucleoli undergo

a series of remarkable changes. In Allantus nucleolar particles pass out into the cytoplasm but not in Thrinax.

In another contribution Gresson (1929) showed that in Thrinax macula and mixta, and Allantus pallipes fatty yolk-bodies arise by deposition of free fat inside the Golgi vesicles. Albuminous yolk bodies arise by the interaction of the nucleolar extrusion upon the cytoplasm.

The works of Stuhlmann (1886), Gross (1903), Henneguy (1904), and Brunelli (1904) are referred to in some detail under Discussion.

Observation

Polistes hebraeus

General structure of ovary.—The anterior part of the ovariole immediately beneath the terminal filament is composed of a mass of indifferent cells, from which arise the nurse-cells and the oocytes proper and a number of small follicular nuclei. Most of the indifferent cells are seen in rosettes, being closely connected by means of spindle bridges—the remains of the preceding oogonial mitoses. In the next zone the oocytes and nurse-cells are clearly differentiated. Soon after, they occur in an alternating series characteristic of the Hymenoptera.

Nurse-cells.—The Golgi bodies of the earliest nurse-cells appear as a few small elements (Plate 1, fig. 1), which gradually increase in number and get concentrated, for the most part, in a perinuclear zone (Plate 1, fig. 4). Ultimately the Golgi bodies are uniformly distributed. They are mostly rod-shaped and crescentic elements.

In the youngest nurse-cells the mitochondria appear as a few fine granules, which rapidly increase in number and become arranged in a dense perinuclear layer which greatly widens in bigger nurse-cells. Eventually, however, the mitochondria are uniformly dispersed.

The inflow of nurse-cell cytoplasm into the oocyte occurs from time to time, and the nurse-cell mitochondria are carried into the ooplasm with the cytoplasmic stream. Curiously enough the cytoplasmic stream has not been found to contain Golgi bodies. Only a small part of the material of the nutritive chamber inflows into the oocyte. The nutritive chamber, as a whole, degenerates and is ultimately cast off.

Follicular epithelium.—The Golgi elements of the follicular cells resemble those of the oocytes. They are indiscriminately distributed all about the nucleus (Plate 1, fig. 5). I have never been able to observe an extrusion of these elements into the ooplasm.

The mitochondria of the follicular cells are fine grains very much similar to those of the nurse-cells, occurring all over the cytoplasm during the early stages (Plate 2, fig. 26). With a powerful oil immersion lens, they are seen at this stage to infiltrate the ooplasm in streams. In more advanced stages the infiltration

ceases, and the mitochondrial granules are found to have gathered up inwards in a juxtanuclear mass, though on the outer side they are still in a more or less evenly dispersed state (Plate 3, fig. 27). In still bigger eggs this mass again disorganizes to give place to the previous condition of uniform dispersal (Plate 3, fig. 30). The most remarkable feature of the follicular cells is the appearance of a number of vesicular bodies at the base, close to the vitelline membrane, which bear a striking resemblance to the early secondary nuclei of the egg proper (Plate 3, fig. 30). It is difficult to make out whether they possess a limiting membrane, and it has not been possible to trace their origin, for they seem to appear all at once. In a few cases the nucleolar particles are found extruded into the cytoplasm of the follicular cells, but this does not appear to be associated with the appearance of these structures. Some of them possess a deeply-stained granule.

The oocyte. Golgi bodies.—The Golgi bodies of the youngest clearly differentiated oocytes are noticed as a few small, osmiophil structures, lying here and there in the cytoplasm (Plate 1, fig. 1). They speedily increase in number, and gather for the most part around the nuclear membrane (Plate 1, fig. 2). Soon afterwards, the Golgi elements are scattered all over the cytoplasm more or less uniformly (Plate 1, fig. 3). In older eggs the greater number of Golgi bodies are concentrated on the periphery (Plate 1, fig. 5). The peripheral accumulation is somewhat accentuated in still bigger eggs. In yolk-laden eggs the Golgi elements are found dispersed more or less uniformly over a broad cortical zone, leaving the interior of the egg almost devoid of them. In mature eggs they are still noticed lying amongst the deutoplasmic spheres (Plate 1, fig. 6).

The Golgi elements of the earliest oocytes are very small structures, but with the general growth of the egg the individual elements experience a considerable degree of growth. Morphologically they resemble crescentic rodlets (Plate 1, figs. 5 and 6). A few rings and platelets may also be observed. In greatly enlarged eggs the chromophobic component can also be easily made out (Plate 1, fig. 6).

The Golgi bodies of the eggs of this species are not easily demonstrable by the silver techniques.

Mitochondria.—In the indifferent elements composing the rosettes the mitochondria are observed as a few minute grains scattered haphazard in the cytoplasm. In the earliest clearly differentiated oocyte the mitochondria are concentrated in a juxtanuclear crescentic cloud (Plate 2, fig. 16), which gradually spreads out to establish a thin perinuclear layer (Plate 2, figs. 17 and 18).

The mitochondria multiply profusely, and the perinuclear mass becomes increasingly wide and dense (Plate 2, figs. 19 and 20). This circumnuclear belt

soon begins to loosen and the granules slowly spread out (Plate 2, fig. 21). Eventually it disappears, and the major portion of the mitochondrial contents spread along the two sides of the oocyte immediately beneath the follicular epithelium (Plate 2, fig. 25); the rest of the cytoplasm is strewn with a few mitochondria. Ultimately the mitochondrial grains arrange themselves completely in a cortical layer extending all round except beneath the nutritive chamber (Plate 2, fig. 26). In more advanced eggs the mitochondrial granules spread inwards, although the granules immediately beneath the follicular epithelium remain in a much greater concentration than elsewhere (Plate 2, figs. 27, 28 and 29).

The mitochondria in this animal are exceedingly fine grains and rodlets. In mature eggs, carrying a considerable deposit of deutoplasmic spheres, it is exceedingly difficult to detect them.

Nucleolar extrusion.—The single nucleolus of the early oocyte gives rise to two or three by its fragmentation, and it is a remarkable fact that even at this stage some of the plasmosomal fragments pass out bodily through the nuclear membrane and lodge themselves in the cytoplasm close to the nucleus (Plate 2, figs. 16, 17, 18, 23 and 24). As the oocyte gradually enlarges, the nucleoli show a high degree of growth and fragmentation, and numerous particles are continuously emitted into the cytoplasm, some of which are remarkably big (Plate 2, figs. 20, 21, 22, 25 and 26). Later on, the extruded nucleolar particles break down into numerous fragments mostly of a small size (Plate 3, figs. 27 and 28). For the most part they are confined to a broad peripheral zone. The nucleolus and nucleolar extrusions are consistently oxyphil (Plate 2, figs. 22, 23 and 24).

Secondary nuclei.—The most noteworthy feature of the oocytes is the origination of the enigmatic structures, the so-called secondary nuclei, at a remarkably early stage (Plate 2, figs. 19 and 20). Some of the minute extruded nucleolar particles situated close to the nuclear membrane get each surrounded by a clear space, and the whole structure presents the appearance of a small vesicle with a single deeply staining granule in the centre.

They gradually increase in size, and simultaneously new ones continue to arise by exactly the same process, all remaining confined to the area immediately surrounding the nucleus (Plate 2, figs. 20, 21 and 22). As illustrated in figs. 20 and 25, Plate 2, the single nucleolus of nearly each of the secondary nuclei is converted into two or three particles by the process of growth and fragmentation, and appearances are suggestive of the extrusion of these particles into the cytoplasm like those of the principal nucleus. For a considerable length of time the secondary nuclei remain surrounding the principal nucleus (Plate 2, fig. 26). They bear a striking resemblance to the principal nucleus appearing, like the latter, as clear vesicles with a limiting membrane and the nucleolar apparatus; for the rest they are optically entirely empty.

Gradually the secondary nuclei begin to move away from the nuclear membrane, and in fairly advanced oocytes, in which the deposition of proteid yolk is about to commence, they are scattered all round the egg in the cortical region (Plate 3, figs. 27 and 28).

Simultaneously, the secondary nuclei have also experienced a change ; they are no longer clear empty vesicles, but, on the other hand, have transformed into denser homogeneously staining bodies, almost of the same consistency as the general cytoplasm. The single deeply staining nucleolus can be still made out in each case.

Another remarkable feature of the oocytes nearly on the threshold of vitellogenesis is the appearance of minute spheroidal bodies at the extreme periphery which resemble empty vacuoles in appearance (Plate 3, fig. 27). They are extremely minute, and it appears as if they arise independently of the nucleolar particles, for as a rule, they exhibit no trace of any distinctly staining bodies in the interior, and for all practical purposes may be regarded as entirely empty. Gradually some of these empty vacuoles enlarge and acquire each a minute nucleolus-like body (Plate 3, fig. 28), and ultimately they can be definitely recognized as the characteristic secondary nuclei, resembling those of the first generation in every detail. The smaller bodies shown immediately beneath the follicular epithelium in this figure are, no doubt, the transformed vacuolar structures of fig. 27, Plate 3. Some of the slightly bigger secondary nuclei possess many nucleolar bodies, five to six in some cases ; the biggest and highly granular ones belong to the first generation.

In highly advanced eggs the accessory nuclei are found to have undergone a noticeable change (Plate 3, figs. 30 and 31). Some of them, as can be observed at once, have attained an enormous size and measure many times the biggest yolk spheres. Each possesses an enclosing membrane and the interior is now completely occupied by a highly complicated reticulum composed of sharply staining strands carrying numerous nucleolar particles along their course. This description applies to all but the smallest ones, although some of the bigger ones are partially empty, due, no doubt, to the mechanical disturbance caused by the microtome knife. Shortly afterwards the limiting membranes dissolve, and the contents of the secondary nuclei are gradually absorbed by the cytoplasm, with the consequence that in the mature egg all traces of these structures are lost. As determined by Feulgen's method, the secondary nuclei contain no chromatin.

Vitellogenesis.—There is no fatty yolk in this animal though in rare cases a few Golgi bodies are found in a greatly swollen state (Plate 1, fig. 6).

Albuminous yolk bodies.—With reference to albuminous yolk bodies it is exceedingly difficult to determine the actual rôle of the mitochondrial grains in their elaboration, for from the inception of growth down to the stage when the

first albuminous yolk granules can be distinctly recognized as such they remain as exceptionally fine dust-like particles. In fact, the study of their morphological characteristics and staining reactions point to their inactivity in yolk formation.

On the other hand, from the very earliest stage of growth onwards the ooplasm carries numerous extruded nucleolar particles, thrown off by the principal nucleus, and also to some extent by the secondary nuclei, as the growth proceeds. Only a few extruded particles in all are used up in forming the secondary nuclei, a vast number of them remaining freely scattered. Just before vitellogenesis the extrusions are scattered on the periphery (Plate 3, figs. 27 and 28), and here they gradually grow up into spherules of proteid yolk (Plate 3, fig. 29). I have not observed a direct transformation of the secondary nuclei into albuminous yolk bodies.

Vespa orientalis

Nurse-cells. Golgi bodies.—The nurse-cell Golgi bodies generally occur in a dispersed state, though a perinuclear concentration is sometimes observed. Ultimately, however, the elements are completely scattered. In their morphology the Golgi elements mostly resemble crescents and rodlets.

Mitochondria.—In the early nurse-cells the mitochondrial granules occur in a dispersed condition, which gradually get arranged in a perinuclear layer to be ultimately scattered all over again.

Both the Golgi elements and the mitochondria are carried into the ooplasm with the nurse-cell cytoplasmic stream. As in Polistes, the nutritive chamber as a whole degenerates and is finally cast off.

Follicular epithelium.—The Golgi bodies of the follicular cells are short granules and rodlets, sometimes markedly crescentic, and more or less like those of the nurse-cells, but much shorter (Plate 1, fig. 9). The mitochondria, likewise, resemble those of the nurse-cells. They are dispersed all over the cytoplasm. The inclusions do not infiltrate the ooplasm.

The oocyte. Golgi bodies.—The Golgi elements of the minute indifferent cellular elements of the ovariole are a few small bodies detected all about the nucleus (Plate 1, fig. 7). In sufficiently differentiated oocytes the cytoplasmic zone immediately surrounding the nucleus is of a denser texture than the neighbouring ooplasm, and it is in this narrow zone that the Golgi bodies occur. At first they are confined to a limited area—one or two crescentic stretches of the cytoplasm, but soon they spread out and establish a perinuclear layer (Plate 1, fig. 7). They grow in number and begin to move away from this position, with the result that in slightly bigger oocytes the entire cytoplasm is thinly strewn with them (Plate 1, fig. 8).

Further on, during the entire period of growth, there is hardly any change in the distribution of the Golgi bodies, except that rarely in some highly advanced oocytes a narrow peripheral zone is almost entirely clear of them. For the most part the Golgi elements appear as crescentic rodlets (Plate 1, fig. 9), though a few rings also may be observed here and there. In mature eggs some Golgi bodies are stuck on the albuminous yolk spherules (Plate 1, fig. 9).

Mitochondria.—In definitely differentiated oocytes the mitochondria appear as a perinuclear cloud (Plate 3, fig. 32). As the egg enlarges, they wander away in all directions to fill up the entire cytoplasm, but for quite a long while a dense, more or less crescentic zone of mitochondrial concentration can be made out occupying a certain area in the immediate vicinity of the nucleus (Plate 3, figs. 33 and 34). Soon the mitochondria begin to drift outwards and a definite peripheral zone of mitochondrial concentration of somewhat uneven girth emerges to view (Plate 4, fig. 35). In still more advanced oocytes the cortical concentration of mitochondria attains its climax, and the interior of the oocyte becomes almost entirely devoid of the inclusions (Plate 4, fig. 36). Later, the mitochondria spread towards the interior the peripheral concentration disappearing (Plate 4, figs. 37 and 38), although the mitochondria are still more thickly scattered at the cortex. In considerably advanced oocytes carrying a large deposit of yolky material the mitochondria can be still detected at the periphery of the oocyte, very fine and feebly staining (Plate 4, fig. 39). The mitochondrial elements appear invariably in the form of tiny spherical granules and short rodlets, often presenting a cloudy appearance if gathered up in dense patches. They do not show any sign of swelling.

Nucleolar extrusion.—The youngest available oocytes are generally found to contain a single plasmosome and a few smaller particles which appear to have originated by its fragmentation. In some oocytes a few particles of the same staining reaction and size are also detected in the ooplasm, which points to the fact that the nucleolar fragments pass into the cytoplasm even at this early stage (Plate 3, fig. 32). For some time afterwards, the nucleolar activity is much feebler, and the amount of extruded nucleolar material much less abundant than happens in Polistes (Plate 3, figs. 32 and 34). Soon, however, the nucleolar activity reaches a high level, and a considerable amount of plasmosomal particles finds its way into the ooplasm (Plate 4, figs. 35 and 36). The plasmosome breaks up into numerous irregular clumps of deeply staining material, which are constantly emitted. The dispersed nucleolar extrusions soon drift to the cortical zone (Plate 4, figs. 36 and 37). They, however, do not occur immediately beneath the follicular epithelium in the mitochondrial layer, but below it. In this subcortical zone they are found to persist till vitellogenesis is about to commence (Plate 4, fig. 38). Meanwhile the extrusions have been fragmenting, so that now they are reduced to minute particles.

The nucleolus and nucleolar extrusions, as observed in Bouin methyl blue eosin preparations, are consistently oxyphil.

Secondary nuclei.—The secondary nuclei in this species appear rather late (Plate 3, fig. 33), although sometimes empty vacuoles are observed in quite early oocytes. The secondary nuclei originate from the extruded nucleolar particles in the way described in Polistes, although the evidence available in this case is not so convincing as in that species. For long they remain associated with the principal nucleus (Plate 3, figs. 33 and 34, and plate 4, fig. 35). In the meantime the secondary nuclei attain a great size and show a high nucleolar activity ; the nucleoli grow in size, fragment, and the particles are emitted into the cytoplasm, as happens in the case of the principal nucleus (Plate 4, fig. 35).

At about the same time some exceedingly small secondary nuclei also arise directly beneath the follicular epithelium from the nucleolar extrusions (Plate 4, fig. 36). Their place of origin is irregularly restricted to certain tracts which, however, are always peripheral. At the time of vitellogenesis a few vacuoles arise in some cases at the periphery (Plate 4, fig. 37). They are found to disappear soon.

The accessory nuclei have a long lease of life. After a certain period they break away from the principal nucleus and wander off to the cortical region, where they are irregularly distributed (Plate 4, fig. 38). And then they undergo a change ; instead of clear vesicles they become dense, opaque, homogeneously staining structures which contain many nucleolar particles.

In yolk-laden oocytes they transform into altogether different-looking bodies. Each has a limiting membrane and contains a highly coiled network carrying numerous deeply-stained particles along its course (Plate 4, fig. 39). Slowly they begin to lose their well-defined contour and are eventually completely dissolved into the cytoplasm. The secondary nuclei contain no chromatin at any stage. Their contents do not show positive response to Feulgen's test.

Vitellogenesis.—There are no fatty yolk bodies in the eggs of this animal at any stage.

Albuminous yolk bodies.—The beginning of the deposition of albuminous yolk bodies is recognized in the subcortical region of oocytes which have attained a certain size. The mitochondria seem to play absolutely no direct rôle in their elaboration, and remain consistently inert.

On the other hand, the rôle of the nucleolar extrusions in their production is abundantly clear, for the first albuminous yolk bodies which can be definitely recognized as such are the direct descendants of the extruded nucleolar bodies. The small albuminous yolk spherule, arising in the restricted subcortical region depicted in fig. 37, Plate 4, are simply the representatives of the extruded nucleolar fragments also gathered in this region during the earlier stages shown in fig. 36, Plate 4. That is, the albuminous yolk bodies arise by a direct

transformation of the extruded plasmosomal fragments. Soon these particles grow much bigger, and the whole cortical region is crammed with yolk spheres of all sizes (Plate 4, fig. 39), leaving an empty area in the middle, denoting the line of the cytoplasmic inflow proceeding from the nutritive chamber.

Later on, the albuminous yolk spherules are found scattered throughout the cytoplasm. The advanced albuminous yolk bodies do not take acid fuchsin and remain yellow, and some of the bigger ones also become vacuolate and granular and slowly dissolve out in the ooplasm.

The accessory nuclei are not directly transformed into yolk bodies.

Scolia quadripustulatus

Nurse-cells.—The Golgi bodies of the nurse-cells are irregular spherical granules which are, at first, dispersed all over the cytoplasm but soon get concentrated in the perinuclear zone and persist there for long (Plate 2, fig. 15). Ultimately they are uniformly dispersed.

The young nurse-cells contain a few scattered mitochondrial granules (Plate 4, fig. 41), which speedily increase and come to lie thickly in the perinuclear zone. In older nurse-cells they are uniformly dispersed. As in Polistes and Vespa, a certain amount of nurse-cell cytoplasm with the inclusions flows into the ooplasm. The nutritive chamber as a whole then degenerates and is finally cast off.

During the early stages the nurse-cells and oocytes are connected by rings (Plate 4, fig. 41). Later these rings disappear.

Follicular epithelium.—The Golgi elements of the follicular cells are like those of the oocyte. For the most part they occur close to the nucleus (Plate 2, fig. 14). The mitochondria, likewise, are mostly restricted to the neighbourhood of the nucleus (Plate 5, fig. 44). The inclusions do not infiltrate the ooplasm.

The oocyte, Golgi bodies.—The Golgi bodies of the indifferent cells are represented by a few osmiophil bodies in the shape of irregular granules and rodlets, scattered close to the nuclear membrane (Plate 1, fig. 10). In the earliest completely differentiated oocyte is perceptible a dense perinuclear zone of cytoplasm which carries Golgi bodies as irregular spherical granules (Plate 1, fig. 10). In slightly bigger oocytes the inclusions are found freely scattered through the entire cytoplasm (Plate 1, fig. 11). Soon, however, the Golgi bodies begin to move out-wards, with the result that shortly afterwards they are entirely gathered up at the periphery (Plate 1, fig. 12). In greatly enlarged oocytes the inclusions still remain aggregated at the periphery. The cortical zone of concentration, however, becomes somewhat broader corresponding to the increased size of the egg, and is not uniform in constitution, for the greater number of the Golgi bodies are

concentrated in the narrow inner zone, while in the outer part only a few elements are thinly scattered (Plate 2, fig. 13). Ultimately, however, the Golgi bodies occupy the entire cytoplasm (Plate 2, fig. 14).

Mitochondria.—The mitochondria of the indifferent cells are difficult to detect. In the early oocytes they appear as numerous fine granules surrounding the nucleus almost from all sides, though for the most part aggregated towards the side where the cytoplasm is more abundant (Plate 4, fig. 40). The mitochondria gradually surround the nucleus from all sides and simultaneously spread out into the cytoplasm (Plate 4, fig. 41) till the entire ooplasm is uniformly filled with them (Plate 4, fig. 42). During the later stages of growth the mitochondria are exclusively concentrated at the periphery (Plate 5, figs. 43). On the side facing the nutritive chamber they are altogether wanting, as they are forced aside by the cytoplasmic stream. In more advanced oocytes the mitochondria spread out towards the interior to some extent (Plate 5, fig. 44). The cortical arrangement of mitochondria is again observable in eggs in which proteid yolk-formation has progressed to an extent (Plate 5, fig. 45), but ultimately the inclusions spread all over the cytoplasm (Plate 5, fig. 46).

Nucleolar extrusion.—The youngest oocyte possesses a single plasmosome and its fragments, and many similar granules are also detected in the ooplasm, which must have originated in the first instance by the fragmentation and extrusion of the plasmosomal particles (Plate 4, figs. 40 and 41). In the later stages the nucleolar extrusions are more abundant (Plate 4, fig. 42). The nucleolar particles move away in all directions in the ooplasm, although the neighbourhood of the nucleus contains a great many of them. In the next stage depicted in fig. 43, Plate 5, these nucleolar particles move away completely to the periphery, where they lie scattered among the mitochondrial grains. The nucleolar apparatus, in spite of repeated fragmentation, attains a considerable size in the bigger oocytes. It is observed to be budding, and the process of extrusion obviously continues. The nucleoli and nucleolar extrusions are always oxyphil as observed in Bouin Mann's methyl blue eosin preparations.

Secondary nuclei.—A few vacuolar bodies may be detected close to the nucleus in some very early oocytes (Plate 4, fig. 40), but they seem to disappear soon (Plate 4, fig. 41). They are not of the nature of secondary nuclei. The secondary nuclei in Scolia begin to arise much later than in Polistes, though the manner of origin is the same. A few of the nucleolar particles near the nucleus become each surrounded by a small empty space ; thus arise some bodies looking like miniature nuclei. Later on, a few secondary nuclei have grown considerably bigger and are scattered all round the egg at the periphery except in the zone immediately beneath the nutritive chamber, and along a limited posterior area (Plate 5, fig. 43). The nucleolar apparatus of these nuclei in each case

attains a proportionately big size, and fragments into many particles, which are finally extruded into the cytoplasm. Some of these small nucleolar particles thrown off by the secondary nuclei of the first generation become surrounded each by a small empty space, and thus arise many smaller accessory nuclei of the second generation. This results in the appearance of many groups, more or less isolated from each other, composed of one big secondary nucleus and numerous nucleolar particles extruded by it some of which are already converted into small accessory nuclei. A great quantity of nucleolar particles are thus found to be scattered on the periphery. The principal nucleus, of course, continues to extrude the plasmosomal fragments, but they cannot, from this stage onwards, be distinguished from those extruded by the secondary nuclei. The secondary nuclei, newly originated, remain confined to their position of origin, so that in the bigger oocytes they form completely discrete groups at the periphery (Plate 5, fig. 44). The fragmentation of the nucleoli and their subsequent extrusion is still in progress, and a few secondary nuclei can also be observed arising in the manner described above. Some of the extruded nucleolar particles grow up into greatly enlarged elements, and are found to move inwards to a certain extent. The smaller nucleolar fragments occupy a more external position.

Nearly all the secondary nuclei move to the anterior side, and are closely aggregated there (Plate 5, fig. 45). The cytoplasmic stream of the nurse-cell has stopped. The secondary nuclei in most cases still possess two nucleolar particles, but it cannot be ascertained if any are extruded into the cytoplasm, on account of the large number of small yolk bodies.

In oocytes filled with yolk bodies the secondary nuclei are scattered all over the periphery among the deutoplasmic spheres. Some of them are still small. They now stain homogeneously and contain a single nucleolus and are enclosed by a membrane (Plate 5, fig. 46). After this they disintegrate and disappear.

The secondary nuclei contain no chromatin.

Vitellogenesis.—There are no fatty yolk bodies in the oocytes of this animal.

Albuminous yolk bodies.—The mitochondria remain granular to the end, and play no visible rôle in the deposition of the albuminous yolk bodies. The proteid yolk bodies arise on the cortex by the direct transformation of the nucleolar extrusions (Plate 5, fig. 45), which are found scattered in a great profusion in this area before vitellogenesis commences (Plate 5, fig. 44). For long the proteid yolk bodies are confined to the periphery, but ultimately they are scattered throughout the cytoplasm (Plate 5, fig. 46).

Discussion

Golgi bodies.—From a study of the cytoplasm of the three species of Hymenoptera, used as material in this investigation, the present writer feels convinced that it

is useless to attempt to reduce the morphology of the Golgi apparatus to a standard form. It is well known that there has been a certain amount of disagreement between Nath and Harvey on this point. Nath and his collaborators find, in nearly all the animals they have examined, that the Golgi apparatus of the oocytes occurs in the form of discrete vesicles consisting of an argentophil or osmiophil cortex and an argentophobic or osmiophobic centre. This form, Nath finds, is related to the formation of fat bodies, which comes about through the deposition of fat in the interior of the vesicles, the "Golgi vacuole" ultimately giving rise to a fully formed fatty spherule. Harvey (1925, 1929 and 1931) and Jägersten (1935), who deny the participation of the Golgi elements in the formation of fat bodies, insist that these inclusions exist in the form of scales or platelets, and not in the form of vesicles. Jägersten goes so far as to say that Nath and his collaborators and several other authors (Sharga, 1928; Gresson, 1929 and 1931; Rai, 1930) have mistaken fat bodies for Golgi elements. Schlottke (1931) also thinks that Nath and Gresson describe fat bodies as Golgi elements. It must, however, be remembered that these authors (Nath, Gresson and others) do not exclusively depend on osmic acid for the demonstration of the Golgi elements, but use silver methods like Da Fano and Cajal as well. And silver preparations do not retain fat bodies. Without going into the details of the question here, it may be pointed out that the structure of the Golgi apparatus may naturally vary in different types of animals and in different types of cells. In Polistes and Vespa the Golgi elements occur predominantly in the form of crescentic rodlets, although a few irregular spheroids and rings also may occur here and there; on the other hand, in Scolia they are found mostly in the form of irregular spherical granules A typically vesicular form, such as is described by Nath, Gresson and others, is not encountered anywhere. This shows that the morphology of the Golgi apparatus may vary in closely related forms. Another remarkable feature is the existence of several types of Golgi elements in the same cell. This is shown in these studies and has also been described before by several authors (Gatenby, 1922; Bhattacharya, 1925; Srivastava and Bhattacharya, 1935). The explanation of this fact does not lie in the assumption of Payne and Jägersten that several different types of inclusions have, from time to time, been offered as Golgi elements. The explanation is that the Golgi elements are polymorphic. It is this variability that has caused several authors to use the term "Golgi substance" rather than the Golgi apparatus for these inclusions. It must, however, be emphasized that the Golgi bodies are structural entities, as their constant shape and growth during the development of an oocyte shows.

It is hardly necessary to discuss the view of Tennant Gardiner and Smith (1931) and Walker and Allen (1929) that a specific Golgi apparatus does not exist at all. It is clear that these authors are not aware of the fact that the Golgi elements

have often been observed intra-vitam (See Bhattacharya, 1942). In eggs they are not generally observed intra-vitam probably because their refractive index is the same as that of the ground cytoplasm in which they occur. This, however, would not justify the conclusion that eggs are devoid of Golgi elements. Mere artefacts cannot assume characteristic shapes. From his personal experience the writer can say that the Golgi apparatus of male germ cells of several types of animals can be seen intra-vitam without the application of any dye at all, and, what is equally significant, the apparatus usually possesses the same form in the living cells as appears in good Ludford preparations. The spermatocytes of the land snail, Vaginula, are fine material for a demonstration of the Golgi apparatus intra-vitam.

The Golgi elements in the oocytes of the insects examined do not occur, at any stage, in the form of a juxtanuclear reticulum and hence the issue raised by Voinov about the correlation of the discrete Golgi dictyosomes and the reticulate form does not arise in this case. The Golgi elements are constantly in the form of discrete bodies. It may be mentioned, however, that my findings do not support Voinov's conclusion that the dictyosomes of the animal eggs, which, according to Voinov, are the active elements of the ooplasm and participate in the synthesis of deutoplasmic material, do not correspond to the Golgi system, which is represented separately by a system of network. Nothing corresponding to this has been noticed in the insects examined during the course of this work. Voinov was somewhat inclined towards accepting Parat's vacuome theory, but in a recent work (1934) he has definitely rejected it.

Mitochondria.—The mitochondria occur, in the eggs of the three insects examined, mostly in the form of minute granules, and sometimes as filaments, as has been described by other authors of insect oogenesis. The filo-réticulaire masses of chondriome described by Hosselet (1931) have not been observed in any cellular element of the insect ovaries examined. Hosselet thinks, like many other French authors, that the Golgi elements have no real existence in the living cells, and that in the fixed cells they are represented sometimes by vacuome and sometimes by chondriome. In support of this Hosselet mentions that typical reticulate bodies closely simulating the Golgi apparatus of other authors are observed in the eggs of the Dipterous insects that he has investigated in preparations made according to specific mitochondrial methods, like Zenker-Helly. Such extraordinary patterns of mitochondria have never been described by any other author in any insect (Nusbaum, 1917; Gatenby, 1920; Nath and collaborators, 1929 and 1930; Govaerts, 1913; Gresson, 1929, 1931 and 1933; Voinov, 1925; Payne, 1932; Srivastava, 1934; Srivastava and Bhattacharya, 1935; Ranade, 1932). The evidence obtained by a study of the oogenesis of Polistes, Vespa, and Scolia do not support Hosselet's conclusions. The present writer feels not much hesitation in saying that Hosselet's conclusions

appear to be based on misobservation. Hosselet's description of the periodical rejuvenation of the chondriome at the expense of the nucleolar material extruded through peculiar intranuclear bridges appears equally fantastic. Just as in the case of filo-réticulaire masses of chondriome, nothing corresponding to this is found in the literature on the cytoplasmic components of insect eggs. It seems almost certain that Hosselet has confused the disintegrating particles of extruded nucleolar material with mitochondria. Nucleolar extrusion is fairly abundant in Polistes, Vespa, and Scolia, but the extruded nucleolar particles are always sharply distinguishable from the mitochondrial granules.

There are a few other authors, besides Hosselet, who have derived mitochondria from the nucleus. According to Saguchi (1932), the mitochondrial granules differentiate in the "vitellogene Masse," a structure which is itself derived from the nucleus. Similarly Heberer (1930), in the eggs of Eucalanus, describes the differentiation of mitochondria in the yolk nucleus. The yolk nucleus itself is formed from extruded nucleolar bodies which condense into a juxtanuclear cap-like body. It seems highly improbable that these views are correct. The yolk nucleus of Balbiani and the "vitellogene Masse" are highly specialized structures which are confined to certain animal eggs, whereas the mitochondria are universally present. Now it is a little hard to believe that a cytoplasmic constituent of universal occurrence originates diversely in different eggs. The Hymenopterous eggs do not have any structure corresponding to the yolk nucleus, and the mitochondria are present at all stages, even in the youngest oocyte.

Nucleolar extrusion.—The occurrence of nucleolar material in the cytoplasm of eggs has been reported from a large range of vertebrates and invertebrates. In the oogenesis of insects it appears to be quite a common phenomenon. In the Hymenopterous material examined the extruded nucleolar particles are exceptionally large in number, and give rise to the secondary nuclei and, ultimately, to proteid yolk. Payne (1932), who investigated the oogenesis of a fairly large number of insects, did not find a single case of nucleolar extrusion. The explanation of this is, probably, to be found in the fact that Payne did not use any acid-containing fixative like Bouin, Zenker and Flemming, which would wash away the cytoplasmic inclusions like mitochondria and Golgi elements and facilitate a study of the nucleolar extrusion. Sometimes it is a somewhat difficult matter to study this phenomenon without using such techniques. The author has been able to observe the process of nucleolar extrusion in nearly all preparations except those meant specifically for the demonstration of the Golgi bodies. There cannot be any doubt about the reality of the process, although granules actually passing through the nuclear membrane cannot be generally met with, which is not surprising in consideration of the fact that the act of passing out

must be a fairly rapid process. Besides, since bodies resembling plasmosomes are actually observed in the ooplasm before the onset of vitellogenesis, the alternative to the passing out of the nucleolar particles is Jägersten's hypothesis that nucleolar bodies similar to those inside the nucleus are elaborated in the cytoplasm as well. Jägersten himself is not quite happy about the matter, as is apparent from his remark that, "Die einzige Schwierigkeit finde ich diesbezüglich darin, dass zwei so verschiedene Gebiete der Zelle wie der Kern und das Cytoplasma Gebilde gleicher Beschaffenheit hervorbringen sollen." To say the least, this view is the more improbable of the two.

Secondary nuclei.—As is well known, the secondary nuclei were first described by Blochmann, though it is abundantly clear now that Blochmann made a mistake regarding their origin, which is not very surprising in consideration of the faulty techniques available in those days. Subsequent researches have failed to corroborate his account of "Knospungsprozess" and their multiplication by self-division. In none of the Hymenopterous species examined by me is there any evidence of a budding of the germinal vesicle, or of a division of the secondary nuclei, and there is certainly nothing answering to his earlier description of "knötchenförmigen Verdichtungen" which gradually grew bigger, separated themselves and assumed a nuclear character. Blochmann himself, however, explained in a later communication (1886) that these "Verdichtungen" represented "kleine, helle, rundliche Gebilde."

Stuhlmann (1886) reported these curious bodies in a number of insects. About the secondary nuclei of Hymenoptera he wrote, "Die Körper welche am Keimbläschen auftreten, hielt Blochmann für Kerne, während ich sie als Dotterkerne deute." There is absolutely no reason for considering the Blochmann nuclei as "Dotterkerne"—apart from all considerations whether they are true nuclei or not. They have absolutely nothing in common with the "Dotterkern" of the spider described by Wittich, and Stuhlmann has certainly been led into a great error in conferring this name on Blochmann's secondary nuclei of the Hymenoptera. The structure and composition of the yolk nucleus is an altogether different matter. But it must be understood that by "Dotterkern" Stuhlmann does not perhaps mean to signify the yolk nucleus, for he writes again, "Ich wiederhole also noch einmal dass ich diese Kerne für Dotterconcretion halte." In some these "Dotterkerne" remain diffuse, in others, they may fuse to form a single big "Dotterkern". Commenting on the Dotterkern Stuhlmann has written, "Der Dotterkern stellt eine Concretion von besonderem, von dem gewöhnlichen Dotter verschiedenem Nährungsmaterial dar, das zu irgendeiner Zeit vom Ei resorbirt wird."

Stuhlmann, it appears, has certainly misunderstood the nature of Blochmann's nuclei, for it can hardly be considered a type of yolk.

On the other hand, he has described somewhat similar bodies of nuclear origin in the eggs of numerous insects belonging to the Lepidoptera, Diptera, Coleoptera and Orthoptera which he calls "Reifungsballen." What these "Reifungsballen" correspond to I cannot definitely decide. They may be dissolved fatty yolk bodies, for Stuhlmann's fixing reagents were calculated to destroy them. But I certainly do not agree with Hegner (1915) when he says, "..... it seems therefore possible that the 'Nebenkerne' of Blochmann, the Reifungsballen of Stuhlmann and the 'maturation spheres' of Wheeler may be homologous..." They are not. Stuhlmann, of course, did meet with the secondary nuclei or the Nebenkerne of Blochmann in the Hymenopterous eggs, but he erroneously considered them "Dotterconcretion." The big "Dotterkern" originating by the fusion of the smaller bodies, resembles, in Hegner's opinion, what he describes as "Keimbahn—determinants." The bodies occurring in other insects are apparently even in Stuhlmann's opinion different, and are differently named—"Reifungsballen." Wheeler's "maturation spheres" in Blatta germanica (1889), in the same way, are not homologous to the Nebenkerne of Blochmann.

I have absolutely no doubt that the secondary nuclei are confined in their distribution entirely to the Hymenopterous eggs ; of this I am convinced by my own investigations of the insect eggs. It is significant that the secondary nuclei have not been discovered by the modern workers of insect oogenesis in any species not belonging to the Hymenoptera. The homology of the bodies discovered in Hemiptera by Will (1884) and Ayers (1884) with Blochmann's secondary nuclei is untenable. No such structures are observed in the Hemipterous insect examined by the present writer (unpublished) or in Dysdercus by Bhandari and Nath (1930). The nuclei discovered by me in the nurse-cell mass of Musca are not secondary nuclei, as I erroneously supposed, but merely follicular nuclei. I withdraw my statement regarding their nature in my previous paper (1934).

Govaerts (1913) reported the occurrence of the secondary nuclei in the Hymenopterous insect that he investigated, but could not determine their origin and function.

As for the view of Will (1884), Ayers (1884), Gross (1903), Korschelt (1889), Brunelli (1904) and Henneguy (1904) that they represent follicular epithelial cells that have wandered into the oocyte, it may be said at once that it is incorrect.

Korschelt obviously had no definite evidence to decide their origin. Nevertheless, he is not inclined to accept their origin from the head nucleus, and writes "....... eine Einwanderung von Epithelzellen in das Ei stattfände." Gross (1903) has as little by way of facts to substantiate his claim. The follicular epithelial nuclei do collect in the nutritive chamber as Gross figures, not, however, by a secondary penetration of the nutritive chamber but merely through the

failure of these elements to move towards the periphery from the indiscriminate scattering of the extreme early stages ; but there is nothing at all in the figure of Gross to demonstrate an inflow of these into the oocytes.

In the insects examined by me, especially in Polistes, this mode of origination is altogether excluded, since the secondary nuclei appear very early. And in no species, and at no stage, have I any reason to suppose that the secondary nuclei represent the nuclei of the follicular cells. Similar bodies discovered in the oocyte and nurse-cells of Musca were held by Korschelt to represent the secondary nuclei, but, according to him, may be "Blutkörperchen" or "parasitischen Organismen."

Loyez's statement regarding their origin (1908) that they are produced by the coagulation of chromatin-containing substances inflowing from the nurse-cells, follicular cells, and the germinal vesicle is also difficult to substantiate, for this conclusion rests on an intangible basis. That the fixing reagents should have such a remarkable effect on the infiltrated substances as to produce such definite pictures simulating true nuclei seems improbable. Nevertheless, it must be admitted that there is some force in the facts she brings forward. Loyez mentions in support of her statement that these structures are "pseudonoyaux" and not true nuclei. She adduces the fact that although the big secondary nuclei resemble the true nuclei in their structure, yet all transitional forms between these and smaller bodies, which seem to be produced by the coagulation of the contents of vacuoles, are found. These empty vacuole-like bodies have also been found by me, and they appear to be the forerunners of the second generation of the secondary nuclei in Polistes and Vespa. Moreover, I have been able to observe vacuole-like bodies, some with a central granule, others without it, in the cells of the follicular epithelium, a phenomenon unrecorded before. And this is in no way connected with the extrusion of any nuclear matter, but is easily explained by assuming the correctness of Loyez's hyphothesis. But I have been able to ascertain that, for the most part, the secondary nuclei are formed by the extruded nucleolar particles. Nearly all except a few of the accessory nuclei of the second generation in Polistes are produced in this way, and many of the second generations too may belong to this rank ; only a few, if any, can arise in the way suggested by Loyez. I am, however, in complete agreement with Loyez with regard to the improbability of their being follicular cells.

In Lasius niger (1911), on the other hand, Loyez reaches a different conclusion. The secondary nuclei in this instance are formed by the chromatinic granules extruded only by the germinal vesicles. About their morphological nature she writes : "Enfin ils ont une structure absolument identique à celle de la vésicule gérminative," and again, "Chaque noyau de Blochmann représente en réalité une vésicule gérminative en miniature." As for the nature of the granules emitted

from the nucleus and developing into secondary nuclei, Loyez thinks that they are chemically modified chromation.

Professor Gatenby (1920) who described the secondary nuclei of Apanteles glomeratus, likewise, expressed the opinion that they were derived from the chromatoid granules extruded by the head nucleus. Mukerji (1930) reported, however, that the contents of the secondary nuclei of Apanteles sp. failed to give positive reaction against Feulgen's test.

A question of fundamental importance thus arises as to whether they are real nuclei, and whether they contain any chromatin.

So far as my observations go, I must state at once that the secondary nuclei do not contain any chromatin, nor do the germinal vesicles at that stage, for the matter of that. In fact, no chromatin is ordinarily detected in somewhat advanced oocytes; they either become exceedingly fine, or are chemically transformed. This happens in most of the animals. If the chromatin of the oocyte itself is reduced to such a state of invisibility as to escape detection it would be idle to look for it in the ooplasm, or in the secondary nuclei.

Buchner (1918), however, considered the originating granules as basichromatin particles. My own work for the most part does not substantiate his views. Nowhere have I discovered any evidence of the infiltration of granules from the nurse-cells that give rise to the secondary nuclei, and I also maintain that this is not conclusively established by Buchner. I have not discovered the presence of these structures either in the nurse-cells or in their nuclei. Nor have I seen them dividing as Buchner figures. In the summary Buchner writes about their function : "Sie stellen eine Hilfseinrichtung für das Ei dar, das durch sie in weitgehenden Masse dezentralisiert wird. Sie spielen für das Eiwachstum und die Dotterbildung die gleiche Rolle wie der Eikern."

Marshall's report on these bodies, which he discovered in the oocytes of Polistes pallipes (1907), leaves the matter undecided as to whether they are true nuclei or not, and he adds nothing about their origin. To the writer it appears exceedingly probable that Peacock and Gresson (1928) did not observe the real secondary nuclei when they reported their presence in the Tenthredinidæ. The nuclei shown in the nutritive chamber are the follicular nuclei entangled amidst the nurse-cells, and their figures do not show the existence of any structure in the ooplasm resembling the secondary nucleus described by Blochmann, Gatenby, Buchner, Loyez and others.

Hegner's figures and description (1915) leave the matter of the origin of these structures largely undecided, but he considers their origin by the immigration of the follicular epithelial cells or by budding from the oocyte nucleus improbable, and reaches the conclusion that in Camponotus they arise from the

material which is given off by the oocyte nucleus and which gets enclosed by a membrane and develops into a nucleus-like body. His figures, however, do not show this extruded material. In Apanteles, likewise, the matter is undecided, while in Rhodites he derives them from chromatin-like granules that may have been emitted by the oocytes, follicular cells, or the nurse-cells.

Hogben (1920), likewise, reported their presence in some Hymenoptera and was inclined to accept Loyez's views with regard to their function.

I have discovered these bodies in the oocytes of all the three Hymenopterous species investigated, and in every case their origin has been traced to the naked granules extruded by the nucleus. These granules, of course, I consider to be plasmosomal in nature. As for the vacuolar bodies, some of which do develop into the secondary nuclei, it is difficult to decide how they originate.

With regard to Buchner's (1918) assertion that the basichromatin material is elaborated in the ground cytoplasm, I may add that my work affords no evidence in support of it. The nucleolar particles (called by Buchner basichromatin particles) are not elaborated in the ooplasm, but appear to come from the nucleus. Then it has been ascertained in this piece of work that the secondary nuclei also, like the head-nucleus, emit nucleolar particles, which in turn may originate other secondary nuclei. This is seen exceedingly well in Scolia in which a gaod many of them arise in this way.

Whether they can be definitely considered as nuclei it is difficult to decide, for there is much to be said in favour of the view that no structure devoid of chromatin can be considered a nucleus. And in the secondary nuclei chromatin has not been convincingly demonstrated so far. But, as Buchner says, they represent a mechanism for yolk-formation—not concerned possibly with growth—and involve a decentralization of the nuclear function.

In Scolia some of the nucleolar particles emitted by the secondary nuclei originated fresh batches of them, but most of them remain naked in the ooplasm together with those emitted by the principal nucleus. And these transform directly into yolk. In the other Hymenopterous species examined all the nucleolar emissions of the secondary nuclei are used up in yolk formation. Thus the function of yolk-formation is discharged, not by the principal nucleus alone, but also by the secondary nuclei. I, however, cannot corroborate Loyez's and Buchner's conclusion, that they directly transform into yolk bodies. Govaerts (1913) rejected this opinion, and Gatenby's work (1920) likewise did not support it. The secondary nuclei simply disintegrate and disappear.

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Description of Plates

Abbreviations used:—*A.Y.*, albuminous yolk; *F.C.*, follicular cell; *G.*, Golgi body; ***M.***, mitochondria; *N.*, principal nucleus; *N.C.*, nurse-cell; ***N.E.***, nucleolar extrusion; *S.N.*, secondary nuclei.

All the diagrams were made with camera lucida, and drawn to the magnification shown on plate 5.

Plate 1

(Polistes hebraeus)

Fig. 1. Young follicle showing Golgi bodies. Ludford.
Fig. 2. Oocyte showing Golgi bodies. Ludford.
Fig. 3. Oocyte showing Golgi bodies and secondary nuclei. Ludford.
Fig. 4. Nurse-cell showing Golgi bodies. Ludford.
Fig. 5. Part of an oocyte showing Golgi bodies. Ludford.
Fig. 6. Part of an oocyte showing Golgi bodies and proteid yolk. Ludford.

(Vespa orientalis)

Figs. 7 & 8. Young oocytes showing Golgi bodies. Ludford.
Fig. 9. Part of an oocyte showing Golgi bodies and albuminous yolk. Ludford.

(Scolia quadripustulatus)

Fig. 10. Oocytes showing Golgi bodies. Ludford.
Fig. 11. Oocyte showing Golgi bodies. Ludford.
Fig. 12. Oocyte showing Golgi bodies and secondary nuclei. Ludford.

Plate 2

Figs. 13 & 14. Parts of oocytes showing Golgi bodies and secondary nuclei. Ludford.
Fig. 15. Nurse-cell showing Golgi bodies. Ludford.

PLATE I

MURLI DHAR—*Cytoplasmic Inclusions in the Oogenesis of Certain Hymenoptera.*

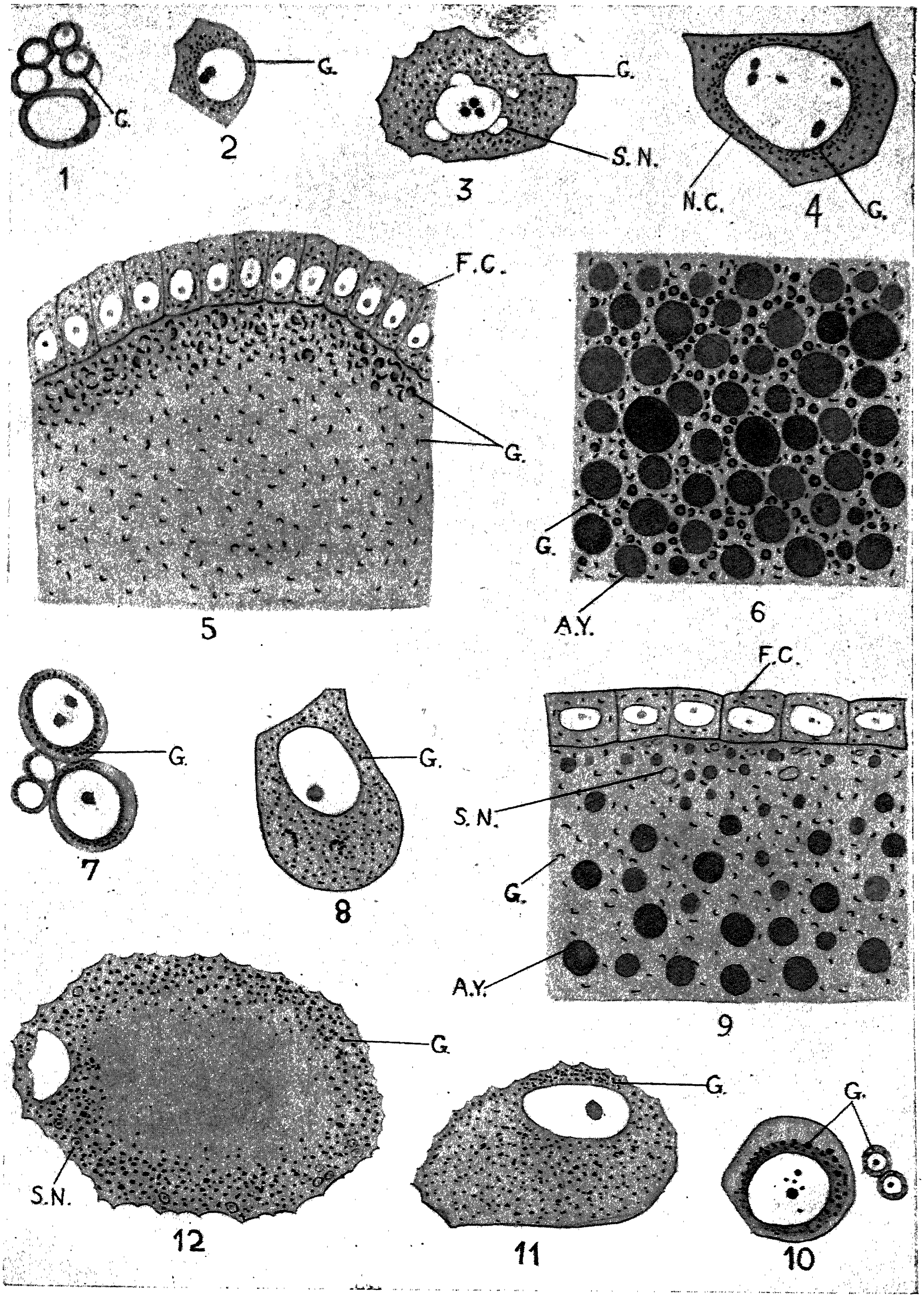


Plate II

Murli Dhar—*Cytoplasmic Inclusions in the Oogenesis of Certain Hymenoptera.*

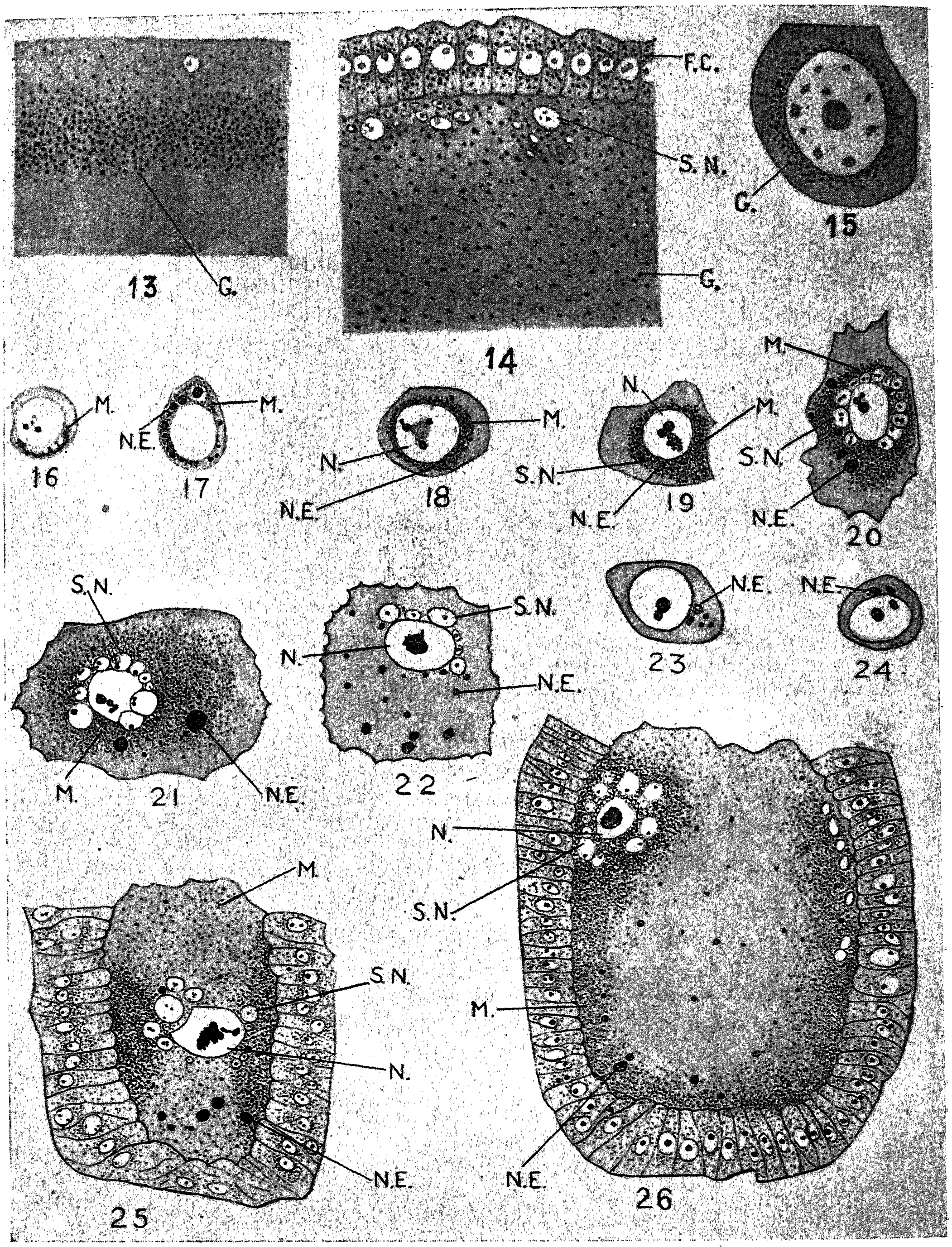
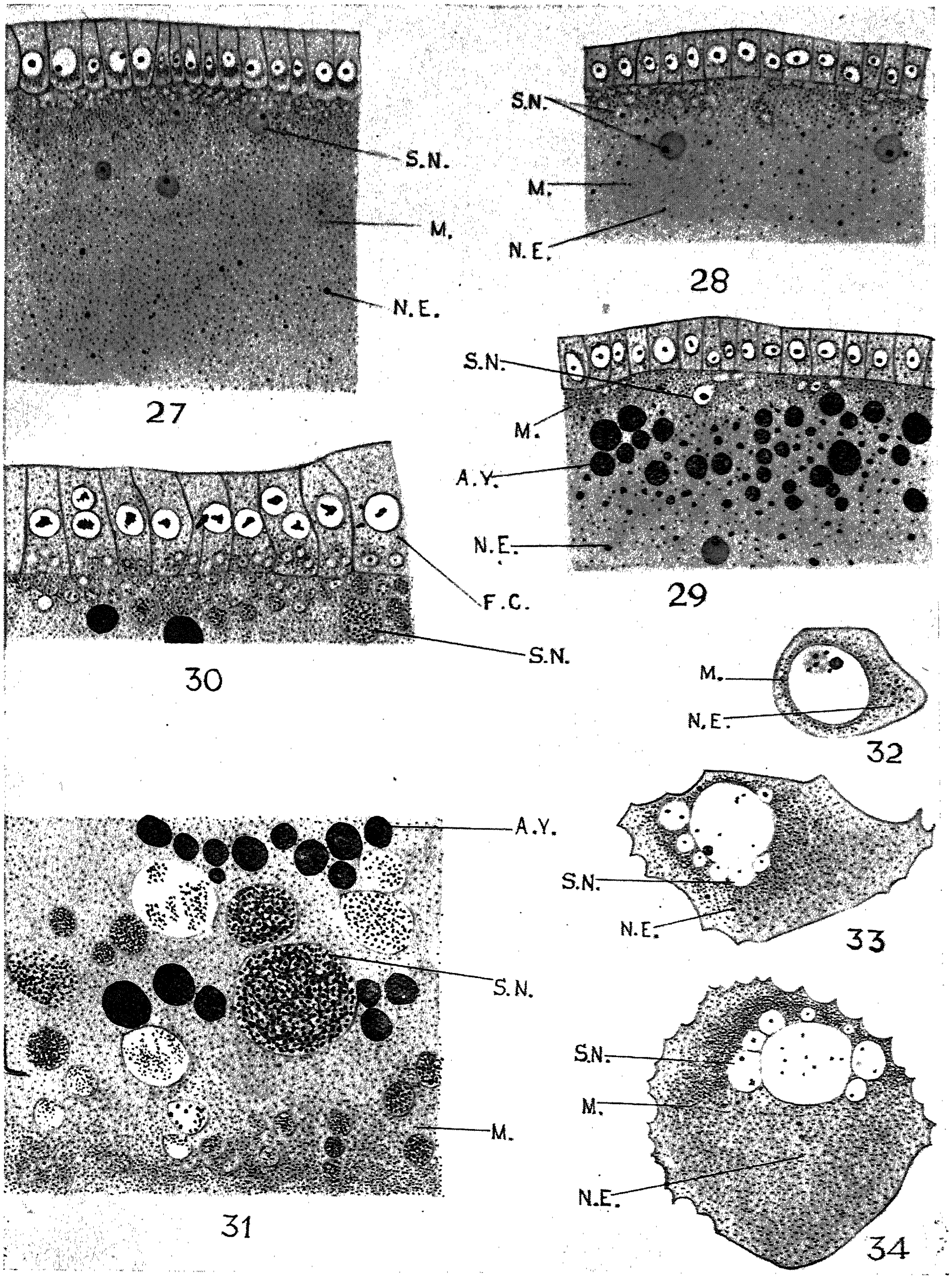


PLATE III

MURLI DHAR—*Cytoplasmic Inclusions in the Oogenesis of Certain Hymenoptera.*



(Polistes hebraeus)

Fig. 16. Young oocyte showing vacuolar bodies and mitochondria. F.W.A., acid fuchsin and methyl green.

Figs. 17&18. Oocytes showing mitochondria and nucleolar extrusions. F.W.A., acid fuchsin and methyl green.

Figs. 19, 20 &21. Oocytes showing mitochondria, nucleolar extrusions, and secondary nuclei. F.W.A.., acid fuchsin and methyl green.

Figs. 22, 23 & 24. Oocytes showing nucleolar extrusions and secondary nuclei. Bouin, Mann's methyl blue eosin.

Figs. 25 & 26. Oocytes showing mitochondria, nucleolar extrusions, and secondary nuclei. F.W.A., acid fuchsin and aurantia.

PLATE 3

(Polistes hebraeus)

Figs. 27 & 28. Parts of egg-periphery showing mitochondria, nucleolar extrusions, and secondary nuclei. F.W.A., acid fuchsin and methyl green.

Fig. 29. Part of egg-periphery showing mitochondria, secondary nuclei and proteid yolk. F.W.A., acid fuchsin and methyl green.

Fig. 30. Part of egg-periphery showing mitochondria, proteid yolk, secondary nuclei, and vesicular bodies resembling secondary nuclei in the follicular cells F.W.A., acid fuchsin and methyl green.

Fig. 31. Part of an oocyte showing mitochondria, secondary nuclei, and proteid yolk. F.W.A , acid fuchsin and methyl green.

(Vespa orientalis)

Fig. 32. Young oocyte showing mitochondria and nucleolar extrusions. Champy-Kull.

Figs. 33 & 34. Oocytes showing mitochondria, nucleolar extrusions, and secondary nuclei. Champy-Kull.

PLATE 4

(Vespa orientalis)

Fig. 35. Oocyte showing mitochondria, nucleolar extrusions, and secondary nuclei. Champy-Kull.

Figs. 36, 37 & 38. Peripheral parts of oocytes showing mitochondria, nucleolar extrusions, and secondary nuclei. Champy-Kull.

Fig. 39. Peripheral part of an oocyte showing mitochondria, secondary nuclei and proteid yolk. Champy-Kull.

(Scolia quadripustulatus)

Fig. 40. Young oocyte showing mitochondria, vacuolar bodies and nucleolar extrusions. F.W.A., acid fuchsin and methyl green.

Fig. 41. Oocyte showing mitochondria and nucleolar extrusions. Two nurse-cells showing mitochondria. F.W.A., acid fuchsin and methyl green.

Fig. 42. Oocyte showing mitochondria, and nucleolar extrusions. F.W.A., acid fuchsin and methyl green.

PLATE 5

(Scolia quadripustulatus)

Fig. 43. Oocyte showing the principal nucleus, the secondary nuclei, mitochondria and nucleolar extrusions F.W.A., acid fuchsin and methyl green.

Fig 44. Part of egg-periphery showing mitochondria, secondary nuclei and nucleolar extrusions. F.W.A., acid fuchsin and methyl green.

Fig. 45. Part of egg-periphery showing mitochondria, secondary nuclei, nucleolar extrusions and proteid yolk. F.W.A., acid fuchsin and methyl green.

Fig 46. Part of egg-periphery showing mitochondria, secondary nuclei and proteid yolk. F.W.A., acid fuchsin and methyl green.

PLATE IV

MURLI DHAR—*Cytoplasmic Inclusions in the Oogenesis of Certain Hymenoptera*

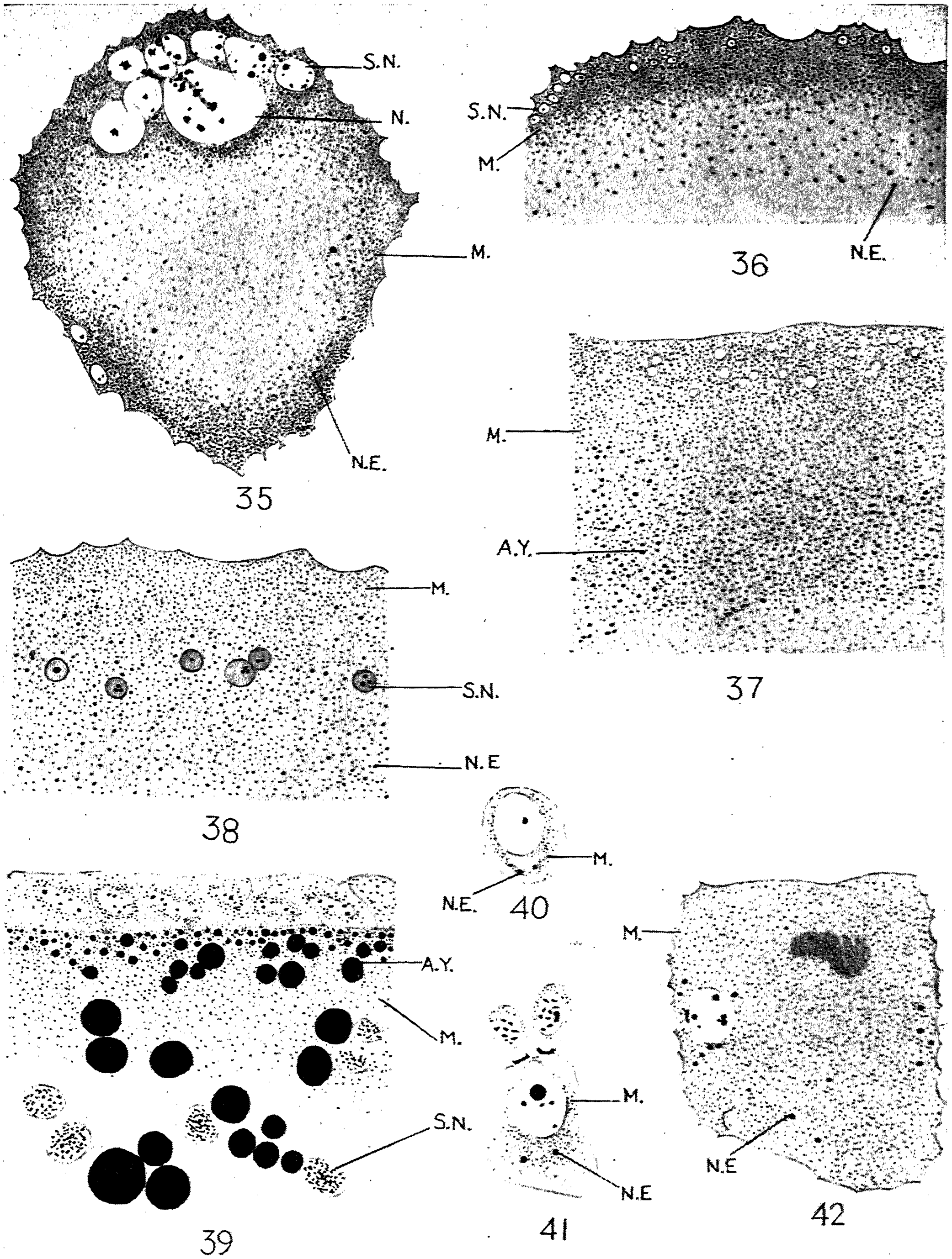
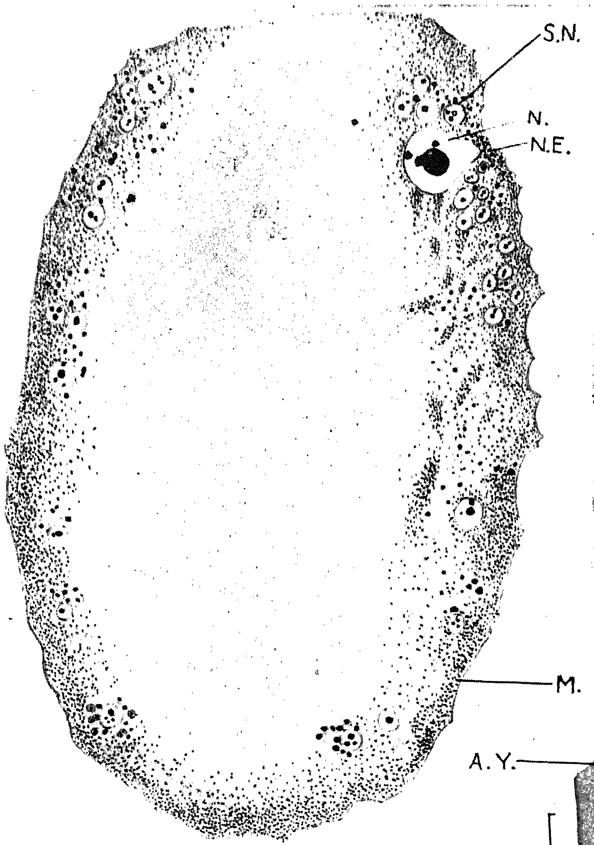
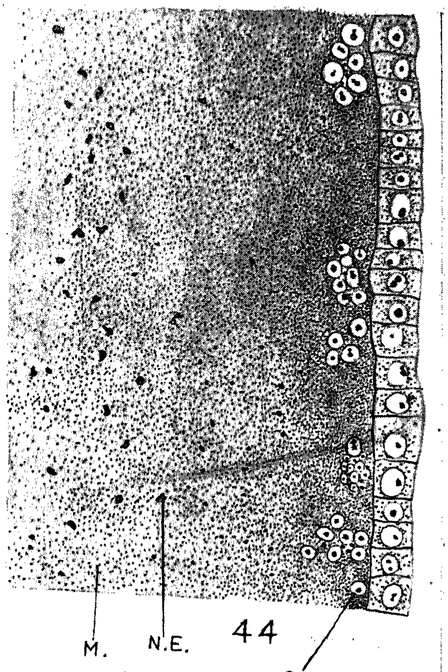


PLATE V

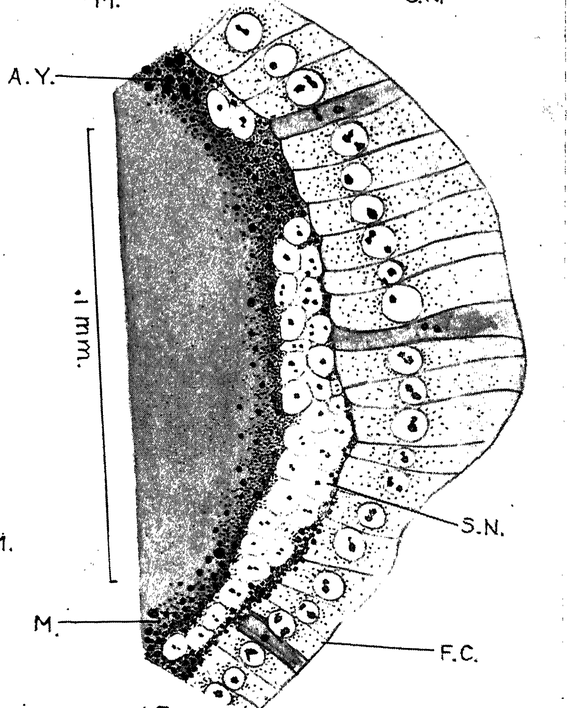
MURLI DHAR—*Cytoplasmic Inclusions in the Oogenesis of Certain Hymenoptera*



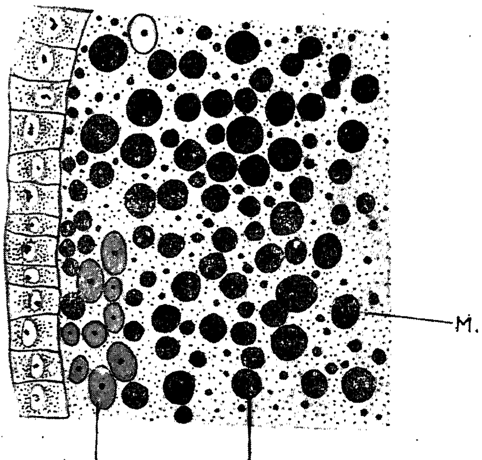
43



44



45



46

PROCEEDINGS

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(SECTION A)

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INFINITE INTEGRALS INVOLVING STRUVE'S FUNCTIONS (II)

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The object of this note is to evaluate some infinite integrals involving Struve's function defined by

$$H_\nu(x)=\sum_0^\infty \frac{(-1)^\gamma\, x^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\,\Gamma(\nu+\gamma+\frac{3}{2})}\text{*} \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Let}\quad I=\int_0^\infty x^{l-1}\, e^{-\frac{1}{2}x^2}\, W_{k,m}(x^2)\, H_\nu(bx)\, dx, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $R(l+\nu\pm 2m)>-2,\ R(\nu)>-1$.

We have

$$I=\int_0^\infty x^{l-1}\, e^{-\frac{1}{2}x^2}\, W_{k,m}(x^2) \sum_0^\infty \frac{(-1)^\gamma\,(bx)^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\,\Gamma(\nu+\gamma+\frac{3}{2})}\, dx \quad . \quad (3)$$

Now, the series (1) is uniformly convergent in any arbitrary interval of values of x for $R(\nu)>-1$. And the function

$$x^{l-1}\, e^{-\frac{1}{2}x^2}\, W_{k,m}(x^2)$$

is continuous. Also, the integral in (2) is absolutely convergent under the conditions imposed. Hence, in (3), we may integrate the series term by term. Thus

$$I=\sum_0^\infty \frac{(-1)^\gamma\, b^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\,\Gamma(\gamma+\frac{3}{2})\,\Gamma(\nu+\gamma+\frac{3}{2})} \int_0^\infty x^{l+\nu+2\gamma}\, e^{-\frac{1}{2}x^2}\, W_{k,m}(x^2)\, dx$$

* G. N. Watson: Theory of Bessel Functions (1922) § 10.4 (2).

Now, we know that

$$\int_0^\infty x^{l-1}\ e^{-\frac{1}{2}x}\,W_{k,m}\ (x)\ dx=\frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)},$$

where $R\,(l\pm m+\frac{1}{2})>0$.*

On using this formula, we find that

$$I=\sum_0^\infty\frac{(-1)^\nu\ b^{\nu+2\gamma+1}\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+\gamma+m+1)\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+\gamma+m+1)}{2^{\nu+2\gamma+2}\ \Gamma(\gamma+\frac{3}{2})\ \Gamma(\nu+\gamma+\frac{3}{2})\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+\gamma-k+\frac{3}{2})}.$$

Thus, we get

$$\int_0^\infty x^{l-1}\ e^{-\frac{1}{2}x^2}\ W_{k,m}\ (x^2)\ H_\nu\ (bx)\ dx$$

$$=\frac{b^{\nu+1}\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+m+1)\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu-m+1)}{2^{\nu+1}\sqrt{\pi}\ \Gamma(\nu+\frac{3}{2})\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu-k+\frac{3}{2})}\times$$

$${}_3F_3\left(\begin{matrix}1,\ \frac{1}{2}l+\frac{1}{2}\nu+m+1,\ \frac{1}{2}l+\frac{1}{2}\nu-m+1\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{1}{2}l+\frac{1}{2}\nu-k+\frac{3}{2}\end{matrix};-\frac{1}{4}\,b^2\right), \quad . \quad . \quad . \quad (4)$$

where $R\,(\nu)>-1$, $R\,(l+\nu\pm 2m)>-2$.

This result is capable of yielding several interesting particular cases.

(*i*) $l=1,\ \ k=m+1,\ \ \nu=2m.$

$$\int_0^\infty e^{-\frac{1}{2}x^2}\ W_{m+1,\,m}\ \ (x^2)\ \ H_{2m}\ \ (bx)\ dx=\frac{b^{2m+1}}{2^{2m+2}}\ e^{-\frac{1}{4}b^2}, \quad . \quad . \quad . \quad (5)$$

where $R\,(m)<\frac{3}{4}$.

On using the formula

$$H_{-\frac{1}{2}}\ (x)=\sqrt{\frac{2}{\pi x}}\ \sin x,$$

we find that for $m=-\frac{1}{4}$, (5) reduces to the well-known result

$$\int_0^\infty xe^{-x^2}\sin bx\ dx=\frac{b\sqrt{\pi}}{4}\ \ e^{-\frac{1}{4}b^2}.$$

---

* S. Goldstein: Operational Representations of Whittaker's Confluent Hyper-geometric Function and Weber's Parabolic Cylinder Function—Proc. London Math. Soc. II 34 (1932), 103—25.

(*ii*) $k=\frac{1}{2}\pm m$.

On using the formula

$$W_{m+\frac{1}{2},\ \pm m}(x)=x^{m+\frac{1}{2}}\ e^{-\frac{1}{2}x},$$

we get

$$\int_0^\infty x^{p-1}\ e^{-x^2}\,H_\nu\,(bx)\,dx=\frac{b^{\nu+1}\,\Gamma(\frac{1}{2}p+\frac{1}{2}\nu+\frac{1}{2})}{2^{\nu+1}\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}\times$$

$$ {}_2F_2\left(\begin{matrix}\frac{1}{2}p+\frac{1}{2}\nu+\frac{1}{2},\ 1\\ \frac{3}{2},\ \nu+\frac{3}{2}\end{matrix}\ ;-\tfrac{1}{4}\ b^2\right),$$

where $R(\nu)>-1$, $R(p+\nu)>-1$.

This is a particular case of a formula proved by me recently.*

(*iii*) $\nu=-\frac{1}{2}$

$$\int_0^\infty x^{l-1}\ e^{-\frac{1}{2}x^2}\ W_{k,\,m}\,(x^2)\sin bx\,dx$$

$$=\tfrac{1}{2}b\cdot\frac{\Gamma(\frac{1}{2}l+m+1)\,\Gamma(\frac{1}{2}l-m+1)}{\Gamma(\frac{1}{2}l-k+\frac{3}{2})}\times$$

$$ {}_2F_2\left(\begin{matrix}\frac{1}{2}l+m+1,\ \frac{1}{2}l-m+1\\ \frac{3}{2},\ \frac{1}{2}l-k+\frac{3}{2}\end{matrix}\ ;-\tfrac{1}{4}b^2\right), \quad \text{. . . . (6)}$$

where $R(l\pm 2m)>-2$.

(*iv*) $\nu=\frac{1}{2}$.

On using the formula

$$H_{\frac{1}{2}}\,(x)=\sqrt{\frac{2}{\pi x}}\ (1-\cos x),$$

we get

$$\int_0^\infty x^{l-1}\ e^{-\frac{1}{2}x^2}\,(1-\cos bx)\ W_{k,m}\,(x^2)\,dx$$

$$=\frac{b^2}{4}\cdot\frac{\Gamma(\frac{1}{2}l+m+\frac{3}{2})\,\Gamma(\frac{1}{2}l-m+\frac{3}{2})}{\Gamma(\frac{1}{2}l-k+2)}\times$$

$$ {}_3F_3\left(\begin{matrix}1,\ \frac{1}{2}l+m+\frac{3}{2},\ \frac{1}{2}l-m+\frac{3}{2}\\ \frac{3}{2},\ 2,\ \frac{1}{2}l-k+2\end{matrix}\ ;-\tfrac{1}{4}b^2\right), \quad \text{. . . . (7)}$$

where $R(\nu)>-1$, $R(l\pm 2m)>-3$.

---

* Infinite Integrals involving Struve's functions (I)—in the press.

(*v*) $m=\pm\frac{1}{4}$.

On using the formula

$$W_{k,\ \pm\frac{1}{4}}\ (x)=2^{-k+\frac{1}{4}}\ x^{\frac{1}{4}}\ D_{2k-\frac{1}{2}}\ (\sqrt{2x}),$$

we get

$$\int_0^\infty x^{l-1}\ e^{-\frac{1}{2}x^2}\ D_k\ (x\sqrt{2})\ H_\nu\ (bx)\ dx$$

$$=\frac{b^{\nu+1}\ \Gamma\ (l+\nu+1)}{2^{l+2\nu-\frac{1}{2}k+1}\ \Gamma(\nu+\frac{3}{2})\ \Gamma(\frac{1}{2}l-\frac{1}{2}k+\frac{1}{2}\nu+1)}\times$$

$${}_3F_3\left(\begin{matrix}1,\ \frac{1}{2}l+\frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}l+\frac{1}{2}\nu+1\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{1}{2}l-\frac{1}{2}k+\frac{1}{2}\nu+1\end{matrix};-\frac{1}{4}b^2\right), \quad \ldots \quad (8)$$

where $R\ (\nu)>-1,\ R\ (l+\nu)>-1$.

(*vi*) $k=0$.

On using the formula

$$W_{0,\ \nu}\ (x)=\sqrt{\frac{x}{\pi}}\ K_\nu\ (\tfrac{1}{2}x),$$

we get

$$\int_0^\infty x^{l-1}\ e^{-\frac{1}{2}x^2}\ K_m\ (\tfrac{1}{2}x^2)\ H_\nu\ (bx)\ dx$$

$$=\frac{b^{\nu+1}\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+m+\frac{1}{2})\ \Gamma(\frac{1}{2}l+\frac{1}{2}\nu-m+\frac{1}{2})}{2^{\nu+1}\ \Gamma\ (\nu+\frac{3}{2})\ \Gamma\ (\frac{1}{2}l+\frac{1}{2}\nu+1)}\times$$

$${}_3F_3\left(\begin{matrix}1,\ \frac{1}{2}l+\frac{1}{2}\nu+m+\frac{1}{2},\ \frac{1}{2}l+\frac{1}{2}\nu-m+\frac{1}{2}\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{1}{2}l+\frac{1}{2}\nu+1\end{matrix};-\frac{1}{4}b^2\right), \quad \ldots \quad (9)$$

where $R\ (\nu)>-1,\ R\ (l+\nu\pm 2m)>-1$.

(*vii*) We know that, when n is a $+ve$ integer,

$$\overset{m}{\underset{n}{L}}\ (x)=\frac{(-1)^n}{\lfloor n}\ x^{-\frac{1}{2}(m+1)}\ e^{\frac{1}{2}x}\ W_{n+\frac{1}{2}m+\frac{1}{2},\ \frac{1}{2}m}\ (x).$$

Using this formula, we get

$$\int_0^\infty x^{p-1}\ e^{-x^2}\ \overset{m}{\underset{n}{L}}\ (x^2)\ H_\nu\ (bx)\ dx$$

$$=\frac{(-1)^n\ b^{\nu+1}\ \Gamma\ (\frac{1}{2}p+\frac{1}{2}\nu+\frac{1}{2})\ \Gamma\ (\frac{1}{2}p+\frac{1}{2}\nu-m+\frac{1}{2})}{\sqrt{\pi}\ 2^{\nu+1}\ \lfloor n\ \Gamma\ (\nu+\frac{3}{2})\ \Gamma\ (\frac{1}{2}p+\frac{1}{2}\nu-m-n+\frac{1}{2})}\times$$

$${}_3F_3\left(\begin{matrix}1,\ \frac{1}{2}p+\frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}p+\frac{1}{2}\nu-m+\frac{1}{2}\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{1}{2}p+\frac{1}{2}\nu-m-n+\frac{1}{2}\end{matrix};-\frac{1}{4}b^2\right), \quad \ldots \quad (10)$$

where $R\ (\nu)>-1,\ R\ (p+\nu)>-1,\ R\ (p+\nu-2m)>-1$.

(*viii*) We know that, when n is a $+ve$ integer,

$$T_m^n(x) = \frac{1}{\lfloor n \, \Gamma(m+n+1)} \, x^{-\frac{1}{2}(m+1)} \, e^{\frac{1}{2}x} \, W_{n+\frac{1}{2}m+\frac{1}{2}, \frac{1}{2}m}(x).$$

On using this formula, we get

$$\int_0^\infty x^{p-1} \, e^{-x^2} \, T_m^n(x^2) \, H_\nu(bx) \, dx$$

$$= \frac{b^{\nu+1} \, \Gamma(\frac{1}{2}p+\frac{1}{2}\nu+\frac{1}{2}) \, \Gamma(\frac{1}{2}p+\frac{1}{2}\nu-m+\frac{1}{2})}{\sqrt{\pi} \, 2^{\nu+1} \lfloor n \, \Gamma(m+n+1) \, \Gamma(\nu+\frac{3}{2}) \, \Gamma(\frac{1}{2}p+\frac{1}{2}\nu-m-n+\frac{1}{2})} \times$$

$${}_3F_3\left(\begin{matrix} 1, \frac{1}{2}p+\frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}p+\frac{1}{2}\nu-m+\frac{1}{2} \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{1}{2}p+\frac{1}{2}\nu-m-n+\frac{1}{2} \end{matrix} ; -\frac{1}{4}b^2\right), \quad . \quad . \quad . \quad (11)$$

where $R(\nu) > -1$, $R(p+\nu) > -1$, $R(p+\nu-2m) > -1$.

(*ix*) $m = \frac{1}{2}$.

We know that

$$W_{n, \frac{1}{2}}(2x) = \Gamma(n+1) \, k_{2n}(x),$$

were $k(x)$ denotes Bateman's Function and $R(n) > -1$.

On using this formula, we get

$$\int_0^\infty x^{l-1} \, e^{-\frac{1}{2}x^2} \, k_{2p}(\tfrac{1}{2}x^2) \, H_\nu(bx) \, dx$$

$$= \frac{b^{\nu+1} \, \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+\frac{3}{2}) \, \Gamma(\frac{1}{2}l+\frac{1}{2}\nu+\frac{1}{2})}{\sqrt{\pi} \, 2^{\nu+1} \, \Gamma(p+1) \, \Gamma(\nu+\frac{3}{2}) \, \Gamma(\frac{1}{2}l+\frac{1}{2}\nu-p+\frac{3}{2})} \times$$

$${}_3F_3\left(\begin{matrix} 1, \frac{1}{2}l+\frac{1}{2}\nu+\frac{3}{2}, \frac{1}{2}l+\frac{1}{2}\nu+\frac{1}{2} \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{1}{2}l+\frac{1}{2}\nu-p+\frac{3}{2} \end{matrix} ; -\frac{1}{4}b^2\right), \quad . \quad . \quad . \quad (12)$$

where $R(\nu) > -1$, $R(l+\nu) > -1$.

UNIFORM RADIAL OSCILLATIONS OF A STAR

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SUMMARY

It has been shown that the only non-rotating star capable of uniform radial oscillation and therefore of preserving the distribution of the polytropic index throughout the oscillation is the homogeneous sphere. The rotating oblate spheroid of small ellipticity has been shown incapable of uniform oscillation with a well-defined period, except when homogeneous. Both the models have been found to oscillate uniformly only in the fundamental mode with the same period. The bearing of this on the origin of the solar system and of the double stars has been touched upon.

It is well known from the investigations of Drs. Kopal,[13] Chandrasekhar[5] and Sir Arthur Eddington that the Cepheid variables are much less centrally condensed than the main-sequence stars. In fact, as Kopal has stated, the δ-Cephei F-5 stars approach the limit of homogeneity. An entirely different line of approach to the problem was suggested by an analysis due to Professor A. C. Banerji[2] and followed up in two papers by the author.[19, 20] The analysis pointed to the probability of radial oscillations being only possible for the homogeneous star. Instability of radial oscillations was found for various laws of density proceeding in direct and inverse powers of the distance from the centre.

For the fundamental mode of oscillation of the homogeneous star, Sterne[17] has obtained the amplitude to be the same fraction of the radial distance throughout the star. The oscillation is therefore a uniform expansion[3] or contraction of the star, and, from the conservation of the density distribution for such expansions, it follows that the star remains homogeneous throughout the oscillation. For other modes of oscillation this is not true, as the amplitude of the oscillation at any point depends on the distance of the point from the centre,[17] and the distribution of the density will vary with the amplitude of the oscillation. This suggests that the probability of stability of radial oscillations is greatest for the homogeneous star and, even for this, the fundamental mode would be the predominant one. This is relevant to the criticism of Shinjo and Jeans[10] that the velocity-curves in the pulsation theory would contain higher modes of incommensurable period, which have not been observed in the regular variables.

In view of the above, it has been thought desirable in this paper to make a general study of the configurations capable of uniform radial oscillation. In a uniform expansion, in which the radius vector to any point is increased k times, the volume of an infinitesimal element at the point is increased k^3 times, and from the conservation of mass it follows that the density at the point decreases k^3 times. Hence it follows that the distribution of the density in the original configuration is unaltered by uniform expansion (or contraction). As, further, the expansion is adiabatic, and therefore along a definite polytropic, it follows, from the extension by N. R. Sen[18] to variable polytropes of Emden's theorem[4] on uniform polytropes, that the polytropic index at any point is conserved by such an expansion.

It has been shown here that, amongst the non-rotating variable polytropes, only the homogeneous sphere is capable of uniform radial oscillation, and therefore, of preserving the distribution of the polytropic index throughout the oscillation. As actual stars have, however, a small amount of rotation and do not on that account preserve a strictly spherical shape, we have also considered the stellar model of an oblate spheroid of small ellipticity ; the strata of equal density are supposed to be similar spheroids. We have shown that the model cannot execute uniform radial oscillations with a well-defined period, except when it is homogeneous.

It is interesting to speculate on the bearing of our result on the vexed problems of the origin of the solar system and of the double stars. We picture an almost homogeneous Cepheid variable rotating with a very small angular velocity. With radiation of energy, the angular velocity increases, till it becomes great enough to cause an appreciable alteration in the homogeneity of the star. The uniform oscillations then are not possible, as shown here. If we grant the very probable suggestion, as explained above, that no other form of oscillation is possible, matter would be thrown out of the parent Cepheid, out of which the planets would condense, and the attraction of a passing star set them moving sidewise. Russell's difficulty,[16] of how such powerful internal forces (throwing out planetary material) could come into play, would no longer stand. Viewed in this light, the analysis fully supports the entirely novel Cepheid theory of the origin of the solar system advanced by Banerji in a very recent paper.[1]

If there be no passing star, the Cepheid would break up by fission into two comparable masses, giving birth to a double-star system. The fissional method is the normal mode of breaking-up of a rotating gas mass with small central condensation, according to Jeans.[7] It is interesting to note that the theory as here put forward is a merger of the fissional theory[8] of Cepheid variation as advocated by Jeans and of the pulsation theory of Shapley and Eddington.[6] The

spectroscopic binaries would form the connecting-link between the Cepheid variables and the double stars. This latter suggestion, according to Jeans,[9] is supported by observation. Kopal[11] finds that the spectroscopic binaries are approximately homogeneous, which should be the case if they are formed from a Cepheid with an almost uniform density distribution. According to Kopal, the central condensation in binaries comes of ageing.

We will now consider the different stellar models. We will, following Eddington,[6] consider only small adiabatic oscillations, and neglect the square of the amplitude.

Model 1. Polytropic gas sphere of variable index.

Let P, ρ and g be the pressure, density and gravity at a point distant ξ from the centre at any instant of time t, and let the suffix zero denote the undisturbed values of these variables. Let

$$\xi=\xi_0\,(1+\xi_1),\ P=P_0\,(1+P_1),\ \rho=\rho_0\,(1+\rho_1). \qquad (1)$$

If the period of the pulsation be $2\pi/n$, then ξ, P and ρ will contain a factor $\cos nt$, as the oscillations are assumed to be small. Further, if they are adiabatic, we have the following[6] equation :

$$P_1=\gamma\rho_1, \qquad (2)$$

where γ is the effective ratio of the specific heats (regarding the matter and enclosed radiation as one system.) From the conservation of mass, we have

$$\rho\xi^2\,d\xi=\rho_0\,\xi_0{}^2\,d\xi_0 \qquad (3)$$

From (1) we have

$$\frac{d\xi}{d\xi_0}=1+\xi_1, \qquad (4)$$

since for uniform oscillation ξ_1 is independent of ξ_0.

From (1), (3) and (4) we have

$$1+\rho_1=\frac{\rho}{\rho_0}=\frac{\xi_0{}^2}{\xi^2}\cdot\frac{d\xi_0}{d\xi}=\frac{1}{(1+\xi_1)^3}=1-3\xi_1, \qquad (5)$$

to the first power of ξ_1.

From (5) we have

$$\rho_1=-3\xi_1. \qquad (6)$$

From (2) and (6) we have

$$P_1=-3\gamma\,\xi_1 \qquad (7)$$

It is to be noted that both P_1 and ρ_1 are independent of ξ_0, since ξ_1 is so. The equation of oscillatory motion is

$$\frac{1}{\rho}\frac{dP}{d\xi} = -g - \frac{d^2\xi}{dt^2}$$
$$= -g + n^2\xi_0\xi_1 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (3) we have

$$\rho d\xi = \rho_0 d\xi_0 \frac{\xi_0^2}{\xi^2} = \rho_0 d\xi_0 (1-2\xi_1) \quad . \quad . \quad . \quad . \quad (9)$$

From (8) and (9) we have

$$\frac{1}{\rho_0}\frac{dP}{d\xi_0} = -(1-2\xi_1)g + n^2\xi_0\xi_1. \quad . \quad . \quad . \quad . \quad (10)$$

Now we have

$$g = \frac{GM}{\xi^2},$$

where M is the mass interior to ξ.

Hence, we have

$$\frac{g}{g_0} = \frac{\xi_0^2}{\xi^2} = 1-2\xi_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

from (1).

We have from (1), (10) and (11)

$$\frac{1}{\rho_0}\frac{d}{d\xi_0}(P_0 + P_0P_1) = -(1-4\xi_1)g_0 + n^2\xi_0\xi_1,$$

which breaks up into the equation of equilibrium

$$\frac{1}{\rho_0}\frac{dP_0}{d\xi_0} = -g_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and the equation of oscillatory motion

$$\frac{1}{\rho_0}\frac{d}{d\xi_0}(P_0P_1) = 4\xi_1 g_0 + n^2\xi_0\xi_1, \quad . \quad . \quad . \quad (13)$$

which by (7) and (12) reduces to

$$[n^2\xi_0 - (3\gamma-4)g_0]\,\xi_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

As ξ_1 is supposed not zero, the equation (14) further reduces to

$$n^2 = (3\gamma-4)\cdot\frac{g_0}{\xi_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which gives the period of oscillation.

F. 2

As for stable oscillations n^2 should be positive, we have from (15) the well-known result that $\gamma > 4/3$.

Further n given by (15) should be independent of ξ_0. Hence, we must have

$$g_0 = \mu\xi_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where μ is a constant.

From (16) we have

$$\mu\xi_0 = g_0 = \frac{GM(\xi_0)}{\xi_0{}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where $M(\xi_0)$ is the mass interior to the sphere of radius ξ_0.

From (17) we have

$$M(\xi_0) = \frac{\mu}{G}\,\xi_0{}^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Also for the mass of shell of radius ξ_0 and thickness $d\xi_0$ we have

$$dM(\xi_0) = 4\pi\xi_0{}^2\rho_0 d\xi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where ρ_0 is the density of the shell.

From (18) and (19) we have

$$\rho_0 = \frac{3\mu}{4\pi G} = \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Thus we arrive at the important conclusion that amongst the non-rotating stars only the homogeneous sphere is capable of uniform oscillation. For the homogeneous sphere of density ρ we have from (15), (16) and (20)

$$n^2 = 4\pi G\rho(\gamma - \tfrac{4}{3}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which is the result obtained by Sterne.[17]

We note that (21) gives only one value of n, that is, only the fundamental mode of oscillation.

We will now consider the rotating model.

Model 2. Oblate spheroid of small ellipticity.

Taking the axis of rotation as Z—axis, the equation of oscillatory motion is, in polar coordinates,

$$\frac{d^2\xi}{dt^2} = \frac{\partial V}{\partial \xi} - \frac{1}{\rho}\frac{\partial P}{\partial \xi} + \omega^2\xi\sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where V, P and ρ are respectively the gravitational potential, pressure and density at an internal point (ξ, θ, ψ). The term $\omega^2\xi\sin^2\theta$ in (22) is due to the rotation.

Let a and $a\,(1-\varepsilon)$ be the major and minor semi-axes of the spheroid, where ε is the ellipticity of the spheroid connected with its meridional eccentricity e by the relation

$$e^2=2\varepsilon \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The angular velocity ω of rotation is supposed to be small, so that we can neglect the square of the ellipticity ε. We further suppose the shells of equal density to be similar spheroids. The particular case of the homogeneous spheroid has been considered by P. L. Bhatnagar in his unpublished thesis for the D.Phil. degree of the Allahabad University.

Neglecting the square of the ellipticity, the polar equation of the surface of the spheroid can be put in the form

$$\xi=a(1-\varepsilon\cos^2\theta) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

We will use Clairaut's results[21] for this model, *viz.*,

(a)[12]

$$\varepsilon\propto v=\frac{\omega^2}{2\pi G\bar{\rho}}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where $\bar{\rho}$ is the mean density ; and

(b)[14]

$$V=\frac{GM}{\xi}+\frac{a^3}{\xi^3}\left(\tfrac{1}{2}\omega^2a^2-\frac{GM\varepsilon}{a}\right)(\cos^2\theta-\tfrac{1}{3}), \qquad . \quad . \quad . \quad (26)$$

where M is the mass of the spheroid and V is the potential at an external point (ξ, θ, ϕ).

We will, as we have said, retain in our equations only the first power of v, as defined in Clairaut's result (a) above, that is, we consider only such slow rotations that we can neglect the effects arising from ω^4 and higher powers of ω.

The equation (3) of the conservation of mass holds for the uniform oscillation of an ellipsoid. This can be shown as follows.

Let a_0, b_0, c_0 be the semi-axes of the ellipsoid through (ξ_0, θ, ϕ), and let these expand to the values a, b, c respectively.

We know that the volume of a thin homoeoid[15] of semi-axes a, b, c is $4\pi\lambda abc$, where

$$\frac{da}{a}=\frac{db}{b}=\frac{dc}{c}=\frac{d\xi}{\xi}=\lambda. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence, if ρ denote the density of the homoeoid and ρ_0 its undisturbed value, we have, from the conservation of mass,

$$a_0b_0c_0\rho_0\frac{d\xi_0}{\xi_0}=abc\rho\frac{d\xi}{\xi}, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

whence we have equation (3) since

$$\frac{a}{a_0}=\frac{b}{b_0}=\frac{c}{c_0}=\frac{\xi}{\xi_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

We can thus assume the equations (1)–(7) to hold in this case too.

Now, as the attraction of a homoeoid at an internal point vanishes, we have for the value of gravity at an *internal* point (ξ, θ, ϕ)

$$g=-\frac{\partial V}{\partial \xi}=\frac{GM(\xi)}{\xi^2}+\frac{3a^3}{\xi^4}\left(\tfrac{1}{2}\omega^2 a^2-\frac{GM(\xi)\varepsilon}{a}\right)(\cos^2\theta-\tfrac{1}{3}), \quad . \quad . \quad . \quad (30)$$

where $M(\xi)$ is the mass of the spheroid through (ξ, θ, ϕ).

From (24) and (30) we have

$$g=\frac{GM(\xi)}{a^2}+\frac{3\cos^2\theta-1}{2}\omega^2 a+\frac{GM(\xi)}{a^2}\varepsilon\sin^2\theta$$

$$=\frac{GM(\xi)}{a^2}\left(1+\frac{3\cos^2\theta-1}{2GM(\xi)}\omega^2 a^3+\varepsilon\sin^2\theta\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

We have a similar equation for g_0 :

$$g_0=\frac{GM(\xi_0)}{a_0^2}\left(1+\frac{3\cos^2\theta-1}{2GM(\xi_0)}\omega^2 a_0^3+\varepsilon\sin^2\theta\right), \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where $M(\xi)=M(\xi_0)$ due to conservation of mass.

From (31) and (32) we have

$$\frac{g}{g_0}=\frac{a_0^2}{a^2}\left[1+\frac{3\cos^2\theta-1}{2GM(\xi_0)}\omega^2(a^3-a_0^3)\right] . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

For uniform oscillation we have

$$a=a_0(1+\xi_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

From (33) and (34) we have

$$g=g_0(1-2\xi_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

From (22) we have the equation of oscillatory motion to be

$$\frac{1}{\rho}\frac{\partial P}{\partial \xi}=-g+n^2\xi_0\xi_1+\omega^2\xi\sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

as ξ_1 in (1) contains a factor $\cos nt$ for small oscillations.

From (9) and (36) we have

$$\frac{1}{\rho_0}\frac{\partial P}{\partial \xi_0}=-(1-2\xi_1)g+n^2\xi_0\xi_1+\omega^2\xi\sin^2\theta. \quad . \quad . \quad . \quad (37)$$

We have from (1), (35) and (37)

$$\frac{1}{\rho_0}\frac{\partial}{\partial \xi_0}(P_0+P_0P_1) = -(1-4\xi_1)\,g_0 + n^2\xi_0\xi_1+\omega^2\xi_0\sin^2\theta, \quad . \quad . \quad (38)$$

which breaks up into the equation of relative equilibrium

$$\frac{1}{\rho_0}\frac{\partial P_0}{\partial \xi_0} = -g_0 + \omega^2\xi_0\sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and the equation of oscillatory motion

$$\frac{1}{\rho_0}\frac{\partial}{\partial \xi_0}(P_0\,P_1) = 4\xi_1 g_0 + n^2\,\xi_0\xi_1, \quad . \quad . \quad . \quad . \quad (40)$$

which by (7) and (39) reduces to

$$[n^2\xi_0-(3\gamma-4)\,g_0]\,\xi_1=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

For stable oscillations we have again that γ must be greater than 4/3.

From (24), (32) and (41) we have, to the order of approximation adopted,

$$n^2=(3\gamma-4)\frac{GM(\xi_0)}{a_0{}^3}.$$

We have omitted the terms of order ε or v in (42) because of the factor ξ_1 in (41).

For a defined period of oscillation, n must be independent of ξ_0 and a_0. In order that this may be so, we must have, from (42),

$$M(\xi_0) = \nu a_0{}^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where ν is a constant.

From (43) we have

$$d\,M(\xi_0) = 3\nu a_0{}^2\,da_0. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

If ρ_0 be the density of the spheroidal shell through (ξ_0, θ, ϕ), we have, from (27),

$$d\,M(\xi_0) = 4\pi a_0\,c_0\,\rho_0\,da_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Also we have

$$c_0 = a_0\,(1-\varepsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where ε is the ellipticity defined in (23).

From (44), (45) and (46), we have, to the order of approximation adopted,

$$\rho_0 = \frac{3\nu}{4\pi} = \text{a constant.} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

We have omitted the term of order ε or v in (47) because of the factor ξ_1 in (41).

We have from (47) that uniform radial oscillation is only possible for the homogeneous spheroid. If the uniform density be ρ, we have, from (42), (43) and (47),

$$n^2 = 4\pi G\rho\,(\gamma - \tfrac{4}{3}). \qquad (48)$$

From (21) and (48) we see that we get only the fundamental mode of uniform oscillation and the same value for the period of oscillation in this mode for the models 1 and 2 considered above. Further, as we have seen, the models must be of uniform density in order that they may be capable of executing uniform oscillations. The superposition of a small amount of rotation does not affect the period of uniform oscillation of a homogeneous spherical star.

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RADIAL OSCILLATIONS OF A SLOWLY ROTATING STAR

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SUMMARY

It has been shown that the superposition of a small amount of rotation does not materially affect the results arrived at for the radial oscillations of a spherical, non-rotating star.

The adiabatic oscillations of a gas sphere have been considered by a number of workers in order to explain stellar variability. Sir Arthur Eddington[4] prepared the field by his classic investigation of the small adiabatic pulsations of the standard model, obtaining the period-density relation of the Cepheids and a theoretical estimate of the period of the fundamental mode of oscillation of δ-Cephei correct to a factor of 2. Later work has mainly been the derivation of the higher modes of oscillation. Prof. A. C. Banerji[1] has been the first to consider the case when the amplitude of the oscillations is large enough for retention of the second order terms. Banerji has shown that the oscillations in this case become unstable, and on this basis has given a most original and fundamental theory of the origin of the solar system.

Actual stars, however, do possess a small amount of rotation, and, on that account, can be more accurately represented by oblate spheroids of small ellipticity. P. L. Bhatnagar, in his unpublished thesis for the D.Phil. degree of the Allahabad University, has considered small oscillations of a homogeneous oblate spheroid of small ellipticity. We have considered the radial oscillations of a slowly rotating spheroid, assuming that the strata of equal density are similar spheroids and remain so throughout the oscillation. These assumptions are quite plausible in the absence of any external disturbing factors. We have shown that both for small and large radial oscillations, the spheroid may be replaced by the *equivalent sphere*, which is constituted by replacing each spheroidal shell by a spherical one just containing it and of the same density. All the results, therefore, that have so far been obtained for the spherical star hold good also for the slowly rotating one. In particular, Banerji's investigation[1] on the origin of the solar system, in which the parent Cepheid has been assumed to be spherical, applies equally well to the slowly rotating Cepheid.

Taking the axis of rotation as Z-axis, the equation of oscillatory motion for a rotating star is, in polar co-ordinates,

$$\frac{d^2\xi}{dt^2}=\frac{\partial V}{\partial \xi}-\frac{1}{\rho}\frac{\partial P}{\partial \xi}+\omega^2\xi\sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where V, P and ρ are respectively the gravitational potential, pressure and density at an internal point (ξ, θ, ϕ), and ω is the angular velocity of rotation. The term $\omega^2\xi\sin^2\theta$ in (1) is due to the rotation.

We will consider only such small rotations that we can neglect effects arising from higher powers of ω than the second, where ω is the angular velocity of rotation.

We will first consider

Case 1. Small Oscillations.

We will neglect terms of the order of $\xi_1{}^2$, v^2 and $v\xi_1$, where ξ_1 is the amplitude of oscillation and

$$v=\omega^2/(2\pi G\bar{\rho}),$$

$\bar{\rho}$ being the mean density of the spheroid.

Let P, ρ and g be the pressure, density and gravity at a point distant ξ from the centre at any instant of time t, and let the suffix zero denote the undisturbed values of these variables. Let

$$\xi=\xi_0(1+\xi_1),\ P=P_0\,(1+P_1),\ \rho=\rho_0(1+\rho_1), \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\xi_1=a_1\cos nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the period

$$\Pi=2\pi/n\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Following Eddington,[4] we will suppose the oscillations to be adiabatic, so that we have the following equation

$$P_1=\gamma\rho_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where γ is the effective ratio of the specific heats, regarding the matter and enclosed radiation as one system.

Let a and $a\,(1-\varepsilon)$ be the major and minor semi-axes of the spheroid, where ε is the ellipticity of the spheroid connected with its meridional eccentricity e by the relation

$$e^2=2\varepsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Neglecting the square of the ellipticity, the polar equation of the surface can be shown to be

$$\xi=a(1-\varepsilon\cos^2\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We will use Clairaut's results[13] for this model, *viz.*,

(a)[9]
$$\varepsilon \propto v = \frac{\omega^2}{2\pi G\bar{\rho}},$$

where $\bar{\rho}$ is the mean density ;
and (b) [10]

$$V = \frac{GM}{\xi} + \frac{a^3}{\xi^3}\left(\tfrac{1}{2}\omega^2 a^2 - \frac{GM\varepsilon}{a}\right)(\cos^2\theta - \tfrac{1}{3}). \quad . \quad . \quad . \quad . \quad (8)$$

where M is the mass of the spheroid, and V the potential at an external point (ξ, θ, ϕ).

Let a_0, a_0, a_0 $(1-\varepsilon)$ be the semi-axes of the spheroid through (ξ_0, θ, ϕ) and let them expand to a, a, a $(1-\varepsilon)$.

We know that the volume of a thin homoeoid[11] of semi-axes a, b, c

$$= 4\pi k\, abc, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\frac{da}{a} = \frac{db}{b} = \frac{dc}{c} = \frac{d\xi}{\xi} = k. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence, if ρ_0 be the undisturbed value of the density of the spheroidal shell through (ξ_0, θ, ϕ), and ρ its density at any other instant, we have the equation of the conservation of mass

$$a_0{}^3(1-\varepsilon)\rho_0 \frac{d\xi_0}{\xi_0} = a^3(1-\varepsilon)\rho\, \frac{d\xi}{\xi}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since the shells are similar, we have

$$\frac{a}{a_0} = \frac{\xi}{\xi_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (11) and (12) we have

$$\xi_0{}^2\rho_0 d\xi_0 = \xi^2 \rho d\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From (2) we have

$$\frac{d\xi}{d\xi_0} = 1 + \xi_1 + \xi_0\xi_1{}', \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where the dash denotes differentiation with respect to ξ_0.

From (2), (13) and (14) we have

$$1 + \rho_1 = \frac{\rho}{\rho_0} = \frac{\xi_0{}^2}{\xi^2} \cdot \frac{d\xi_0}{d\xi} = 1 - 3\xi_1 - \xi_0\xi_1{}',$$

whence
$$\rho_1 = -3\xi_1 - \xi_0\xi_1{}', \quad . \quad . \quad . \quad . \quad . \quad (15)$$

to the first power of ξ_1.

F. 3

From (5) and (15) we have

$$P_1 = -3\gamma\xi_1 - \gamma\xi_0\xi_1'. \qquad (16)$$

From (1), (2) and (3) we have the equation of oscillatory motion

$$\frac{1}{\rho}\ \frac{\partial P}{\partial \xi} = -g + \omega^2\xi_0 \sin^2\theta - \frac{d^2\xi}{dt^2} \qquad (16^1)$$

$$= -g + \omega^2\xi_0 \sin^2\theta + n^2\xi_0\xi_1 \qquad (17)$$

As the attraction of a homoeoid at an internal point is zero, we have from equation (8) for the value of gravity at an internal point (ξ, θ, ϕ),

$$g = -\frac{\partial V}{\partial \xi} = \frac{GM(\xi)}{\xi^2} + \frac{3a^3}{\xi^4}\left(\tfrac{1}{2}\omega^2 a^2 - \frac{GM(\xi)}{a}\varepsilon\right)(\cos^2\theta - \tfrac{1}{3}), \qquad (18)$$

where $M(\xi)$ is the mass of the spheroid through (ξ, θ, ϕ).

From (7) and (18) we have

$$g = \frac{GM(\xi)}{a^2} + \frac{3\cos^2\theta - 1}{2}\omega^2 a + \frac{GM(\xi)}{a^2}\varepsilon \sin^2\theta$$

$$= \frac{GM(\xi)}{a^2}\left(1 + \frac{3\cos^2\theta - 1}{2\,GM(\xi)}\ \omega^2 a^3 + \varepsilon \sin^2\theta\right). \qquad (19)$$

We have a similar equation for g_0 :

$$g_0 = \frac{GM(\xi_0)}{a_0^2}\left(1 + \frac{3\cos^2\theta - 1}{2\,GM(\xi_0)}\omega^2 a_0^3 + \varepsilon \sin^2\theta\right). \qquad (20)$$

Now for conservation of mass $M(\xi) = M(\xi_0)$. Hence, we have from (19) and (20)

$$\frac{g}{g_0} = \frac{a_0^2}{a^2}\left[1 + \frac{3\cos^2\theta - 1}{2\,GM(\xi_0)}\omega^2 (a^3 - a_0^3)\right] \qquad (21)$$

As we have assumed that the spheroidal form is preserved in the oscillation, we have

$$\frac{a}{a_0} = \frac{\xi}{\xi_0} = 1 + \xi_1, \qquad (22)$$

from (2).

From (21) and (22), we have

$$g = g_0(1 - 2\xi_1). \qquad (23)$$

From (13) we have

$$\rho d\xi = \rho_0 \frac{\xi_0^2}{\xi^2} d\xi_0 = \rho_0(1 - 2\xi_1)d\xi_0 \qquad (24)$$

From (17) and (24) we have

$$\frac{1}{\rho_0}\frac{\partial P}{\partial \xi_0} = -(1-2\xi_1)\,g + n^2\xi_0\xi_1 + \omega^2\xi_0\sin^2\theta \quad . \quad . \quad (25)$$

We have from (2), (23) and (25)

$$\frac{1}{\rho_0}\frac{\partial}{\partial \xi_0}(P_0 + P_0P_1) = -(1-4\xi_1)\,g_0 + n^2\xi_0\xi_1 + \omega^2\xi_0\sin^2\theta \quad . \quad . \quad (26)$$

Equation (26) breaks up into the equation of relative equilibrium

$$\frac{1}{\rho_0}\frac{\partial P_0}{\partial \xi_0} = -g_0 + \omega^2\xi_0\sin^2\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and the equation of oscillatory motion

$$\frac{1}{\rho_0}\frac{\partial}{\partial \xi_0}(P_0P_1) = 4\xi_1 g_0 + n^2\xi_0\xi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

By (16) and (27), equation (28) reduces to

$$\xi_1'' + (4-\nu)\frac{\xi_1'}{\xi_0} + \left[\frac{n^2}{\gamma}\cdot\frac{\rho_0}{P_0} - \frac{\alpha\nu}{\xi_0^2}\right]\xi_1 = 0, \quad . \quad . \quad . \quad (29)$$

where $\nu = \dfrac{g_0\rho_0\xi_0}{P_0}$ and $\alpha = 3 - \dfrac{4}{\gamma}$ . . . . (30)

Equation (29) is the same as that obtained by Eddington[5] for the small adiabatic oscillations of a gas sphere. The variables ρ_0, P_0 and g_0, however, denote here the undisturbed values for the spheroid.

To the order of approximation adopted, we will omit terms of order v, as defined in Clairaut's result (a) above-mentioned, from g_0 and P_0 occurring in equation (29). To this order of approximation, we have from (20)

$$g_0 = \frac{GM}{a_0^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where M is the mass of the spheroid of semi-major axis a_0 through (ξ_0, θ, ϕ).

From (9) and (10) we have the mass of a thin spheroidal shell of semi-major axis a_0

$$= 4\pi a_0^3\,(1-\varepsilon)\,\rho_0\frac{da_0}{a_0} = 4\pi\rho_0\,(1-\varepsilon)\,a_0^2\,da_0 \quad . \quad . \quad . \quad . \quad . \quad (32)$$

From (32) we have the mass of the spheroid

$$M = (1-\varepsilon)\int 4\pi a_0^2\rho_0 da_0 = (1-\varepsilon)\,M', \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where M′ is the mass of the *equivalent sphere*, which we have already defined in the second paragraph of this paper as a sphere constructed by replacing each spheroidal shell by a spherical one just containing it and of the same density.

Hence, we have from (31), to the degree of approximation indicated above,

$$g_0 = \frac{GM'}{a_0^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where M′ is the mass of the *equivalent sphere.*

From (34) we have

$$g_0 = g_0', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where g_0' is the value of gravity at (ξ_0, θ, ϕ) for the *equivalent sphere.*

To the same degree of approximation, we have from (27)

$$P_0 = \int_{\xi_0}^{r_0} g_0 \rho_0 d\xi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where the radius vector through ξ_0 meets the surface of the spheroid at r_0.

If P_0' denote the corresponding pressure for the *equivalent sphere,* we have

$$P_0' = \int_{\xi_0}^{R_0} g_0 \rho_0 d\xi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where R_0 is the radius of the circumscribing sphere.

We have from (7)

$$r_0 = R_0 (1 - \varepsilon \cos^2\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Let $$\int g_0 \rho_0 d\xi_0 = f(\xi_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Then we have from equations (36)—(39)

$$\begin{aligned} P_0 &= f(r_0) - f(\xi_0) = f[R_0 (1 - \varepsilon \cos^2\theta)] - f(\xi_0) \\ &= f(R_0) - f(\xi_0) \text{ (neglecting terms of order } v) \\ &= P_0' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40) \end{aligned}$$

From (35) and (40) we see that the equation (29) will be unaltered if we replace the spheroid by the *equivalent sphere.*

As (29) is the fundamental differential equation for small adiabatic oscillations, we have the important result that for such oscillations an oblate spheroid of small ellipticity may be replaced by the *equivalent sphere.* In other words, the superposition of a small amount of rotation does not materially alter the results arrived at for small oscillations of the spherical star.

We will next consider

Case 2. Large oscillations,

so that we retain terms of the order of ξ_1^2.

We will at first consider such small rotations that we can neglect $v\xi_1$ and v^2, that is, the effects arising from ω^3, ω^4 and higher powers of ω.

The analogous case for the non-rotating star has been considered[1] by Banerji, and by Bhatnagar in his unpublished thesis for the D.Phil. degree of the Allahabad University. We will show that Banerji's results and therefore his theory of the origin of the solar system hold good for the slowly rotating star.

For adiabatic oscillations, we have

$$P \propto \rho^{\gamma}, \qquad (41)$$

where γ is the effective ratio of the specific heats, regarding the matter and enclosed radiation as one system.

From (2), (13) and (41) we have

$$1+\rho_1 = \frac{1}{(1+\xi_1)^2(1+\xi_1+\xi_0\xi_1')}, \qquad (42)$$

and

$$1+P_1 = \frac{1}{(1+\xi_1)^{2\gamma}(1+\xi_1+\xi_0\xi_1')^{\gamma}} \qquad (43)$$

From (21) and (22) we have

$$g = \frac{g_0}{(1+\xi_1)^2}, \qquad (44)$$

neglecting terms of the order of $v\xi_1$.

From (13) we have

$$\rho d\xi = \rho_0 \frac{\xi_0^{\,2}}{\xi^2} d\xi_0 = \frac{\rho_0 d\xi_0}{(1+\xi_1)^2} \qquad (45)$$

From (16′) and (44) we have

$$\frac{d^2\xi}{dt^2} = -\frac{g_0}{(1+\xi_1)^2} + \omega^2\xi_0 \sin^2\theta + \frac{(1+\xi_1)^2}{\rho_0}\frac{\partial P}{\partial \xi_0}. \qquad (46)$$

It should be noted that in using (16′) to derive (46) we have neglected terms of the order of $v\xi_1$.

(46) by (2) breaks up into the equation of relative equilibrium (27) and the equation of oscillatory motion

$$\frac{d^2\xi}{dt^2} = g_0\left[1-\frac{1}{(1+\xi_1)^2}\right] - \frac{2\xi_1+\xi_1^{\,2}}{\rho_0}\frac{\partial P_0}{\partial \xi_0} - \frac{(1+\xi_1)^2}{\rho_0}\frac{\partial}{\partial \xi_0}(P_0 P_1) \qquad (47)$$

(47) by (27) and (43) reduces to

$$\ddot{\xi} = -g_0\left[\frac{1}{(1+\xi_1)^2} - \frac{1}{(1+\xi_1)^{2\gamma-2}(1+\xi_1+\xi_0\xi_1')^{\gamma}}\right] - \frac{P_0}{\rho_0}(1+\xi_1)^2 \frac{d}{d\xi_0}\left[\frac{1}{(1+\xi_1)^{2\gamma}(1+\xi_1+\xi_0\xi_1')^{\gamma}}\right], \qquad (48)$$

neglecting terms of order $v\xi_1$.

Retaining terms of the order of $\xi_1{}^2$, the equation (48) reduces to

$$\ddot{\xi}=g_0\Big[\{-(3\gamma-4)\xi_1-\gamma\xi_0\xi_1{}'\}+\{(\tfrac{9}{2}\gamma^2-\tfrac{9}{2}\gamma-2)\xi_1{}^2+\gamma(3\gamma-1)\xi_0\xi_1\xi_1{}'$$

$$+\frac{\gamma(\gamma+1)}{2}\xi_0{}^2\xi_1{}'^2\}\Big]+\frac{P_0\gamma}{\rho_0}\,[(4\xi_1{}'+\xi_0\xi_1{}'')-\{4(3\gamma-1)\xi_1\xi_1{}'$$

$$+2(2\gamma+1)\xi_0\xi_1{}'^2+(3\gamma-1)\xi_0\xi_1\xi_1{}''+(\gamma+1)\xi_0{}^2\xi_1{}'\,\xi_1{}''\}] \quad . \quad . \quad . \quad . \quad (49)$$

Eddington[6] has shown that if the square of the amplitude be retained, the complete formula for ξ_1 will be of the form

$$\xi_1=a_1\cos nt-a_2\cos 2\,nt,$$

where a_2 is of order $a_1{}^2$. This would give "a velocity-curve having the general characteristics of the observed velocity-curves of Cepheids, *viz.*, a sharp decrease from maximum to minimum receding velocity and a slower return to maximum with indications of a hump in the curve."[6]

Substituting (50) in (49) and equating the coefficients of $\cos nt$ and $\cos 2nt$ separately to zero, we find that certain terms of the order of $\xi_1{}^2$ are left over. We therefore assume the following form for ξ_1 :

$$\xi_1=a_1\cos nt-a_2\cos 2nt-a_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where a_2 and a_3 are of order $a_1{}^2$.

From (2) and (51) we have

$$\ddot{\xi} = -n^2\,\xi_0\,(a_1\cos nt-4a_2\cos 2\,nt), \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\xi_1{}' = a_1{}'\cos nt-a_2{}'\cos 2nt-a_3{}', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and

$$\xi_1{}'' = a_1{}''\cos nt-a_2{}''\cos 2nt-a_3{}'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where the dots and the dashes denote differentiation respectively with respect to t and ξ_0.

Substituting from equations (51) to (54), equation (49) reduces to

$$\Big[n^2a_1\xi_0+g_0\{-(3\gamma-4)a_1-\gamma\xi_0a_1{}'\}+\frac{P_0\gamma}{\rho_0}\{4a_1{}'+\xi_0a_1{}''\}\Big]\cos nt$$

$$+[-4n^2\xi_0a_2+g_0\{(3\gamma-4)a_2+\gamma\xi_0a_2{}'+\tfrac{1}{4}(3\gamma-4)(3\gamma+1)a_1{}^2+\tfrac{1}{2}\gamma(3\gamma-1)\xi_0a_1a_1{}'$$

$$+\tfrac{1}{4}\gamma(\gamma+1)\xi_0{}^2a_1{}'^2\}+\frac{P_0\gamma}{\rho_0}\{-4a_2{}'-\xi_0a_2{}''-2(3\gamma-1)a_1a_1{}'-(2\gamma+1)\xi_0a_1{}'^2$$

$$-\tfrac{1}{2}(3\gamma-1)\xi_0a_1a_1{}''-\tfrac{1}{2}(\gamma+1)\xi_0{}^2a_1{}'\,a_1{}''\}]\cos 2nt+[g_0\{(3\gamma-4)a_3+\gamma\xi_0a_3{}'$$

$$+\tfrac{1}{4}(3\gamma-4)(3\gamma+1)a_1{}^2+\tfrac{1}{2}\gamma(3\gamma-1)\xi_0a_1a_1{}'+\tfrac{1}{4}\gamma(\gamma+1)\xi_0{}^2a_1{}'^2\}+\frac{P_0\gamma}{\rho_0}\{-4a_3{}'$$

$$-\xi_0a_3{}''-2(3\gamma-1)a_1a_1{}'-(2\gamma+1)\xi_0a_1{}'^2-\tfrac{1}{2}(3\gamma-1)\xi_0a_1a_1{}''-\tfrac{1}{2}(\gamma+1)\xi_0{}^2a_1{}'a_1{}''\}]$$

$$=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Equating the coefficients of cos nt, cos $2nt$ and the remaining terms separately to zero, we have the following equations of oscillatory motion :

$$a_1'' + \frac{4-\nu}{\xi_0} a_1' + \left[\frac{n^2 \rho_0}{P_0 \gamma} - \frac{\alpha\nu}{\xi_0^2} \right] a_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$a_2'' + \frac{4-\nu}{\xi_0} a_2' + \left[\frac{4n^2 \rho_0}{P_0 \gamma} - \frac{\alpha\nu}{\xi_0^2} \right] a_2 = A_1 \quad . \quad . \quad . \quad . \quad . \quad (57)$$

and

$$a_3'' + \frac{4-\nu}{\xi_0} a_3' - \frac{\alpha\nu}{\xi_0^2} a_3 \qquad = A_1, \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where

$$\nu = \frac{g_0 \rho_0 \xi_0}{P_0}, \qquad \alpha = 3 - \frac{4}{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and

$$A_1 = \left[\tfrac{1}{2}(3\gamma-1)\frac{n^2\rho_0}{P_0\gamma} - \tfrac{3}{4}(\gamma-1)\frac{\alpha\nu}{\xi_0^2} \right] a_1^2 + \tfrac{1}{2}(\gamma+1)\left[\frac{n^2\rho_0}{P_0\gamma} - \frac{\alpha\nu}{\xi_0^2} \right] \xi_0 a_1 a_1'$$
$$+ [1 - \tfrac{1}{4}(\gamma+1)\nu] a_1'^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

Equations (56)—(58) are precisely those obtained by Banerji[2] in his investigation on the large radial oscillations of a Cepheid. The variables P_0 and ν here of course refer to the spheroid, but it has been shown in Case 1 that, to the order of approximation adopted, they remain unaltered for the *equivalent sphere.*

We have thus proved that both for small and large radial oscillations, the oblate spheroid of small ellipticity may be replaced by the *equivalent sphere.* The importance of the result arrived at would be realised from the fact that all the investigations carried so far on the radial oscillations of a sphere are seen to apply equally well to the slowly rotating star. In particular, Banerji's attractive theory[1] on the origin of the solar system holds *in toto* for the rotating Cepheid. Further, a good deal of simplification has been effected in all future investigations, which may henceforth be confined, without material prejudice to the results, to the spherical star.

It should be noted that the investigation in Case 1, where we have neglected effects arising from ω^4, is of a higher degree of approximation than that in Case 2, where we have neglected $v\xi_1$, that is, effects arising from ω^3. We will, therefore, now reconsider Case 2, retaining terms of the order of $v\xi_1$. We will only indicate the steps of departure from the investigation already given.

From (21) and (22) we now have

$$g=\frac{g_0}{(1+\xi_1)^2}\left[1+\frac{3\mu^2-1}{2GM(\xi_0)}\omega^2 a_0\xi_1(a^2+aa_0+a_0^2)\right]$$

$$=\frac{g_0}{(1+\xi_1)^2}+\frac{3\mu^2-1}{2GM(\xi_0)}g_0 a_0(a^2+aa_0+a_0^2)\omega^2\xi_1, \quad . \quad . \quad (44')$$

where $\mu=\cos\theta$.

From (1) and (2) we have the equation of oscillatory motion

$$\frac{1}{\rho}\frac{\partial P}{\partial\xi}=-g+\omega^2\xi_0\sin^2\theta+\omega^2\xi_0\xi_1\sin^2\theta-\frac{d^2\xi}{dt^2} \quad . \quad . \quad (16'')$$

From (16″) and (44′) we have

$$\frac{d^2\xi}{dt^2}=-\frac{g_0}{(1+\xi_1)^2}-\frac{(1+\xi_1)^2}{\rho_0}\frac{\partial P}{\partial\xi_0}+\omega^2\xi_0\sin^2\theta+\omega^2\xi_0\xi_1\sin^2\theta$$
$$-\frac{3\mu^2-1}{2GM(\xi_0)}g_0 a_0(a^2+aa_0+a_0^2)\omega^2\xi_1. \quad . \quad . \quad . \quad . \quad (46')$$

(46′) by (2) breaks up into the equation of relative equilibrium (27) and the equation of oscillatory motion

$$\frac{d^2\xi}{dt^2}=g_0\left[1-\frac{1}{(1+\xi_1)^2}\right]-\frac{2\xi_1+\xi_1^2}{\rho_0}\frac{\partial P_0}{\partial\xi_0}-\frac{(1+\xi_1)^2}{\rho_0}\frac{\partial}{\partial\xi_0}(P_0P_1)$$
$$+\omega^2\xi_0\xi_1\sin^2\theta-\frac{3\mu^2-1}{2GM(\xi_0)}g_0a_0(a^2+aa_0+a_0^2)\omega^2\xi_1. \quad . \quad . \quad (47')$$

(47′) by (27) and (43) reduces to

$$\frac{d^2\xi}{dt^2}=-g_0\left[\frac{1}{(1+\xi_1)^2}-\frac{1}{(1+\xi_1)^{2\gamma-2}(1+\xi_1+\xi_0\xi_1')^\gamma}\right]$$
$$-\frac{P_0}{\rho_0}(1+\xi_1)^2\frac{d}{d\xi_0}\left[\frac{1}{(1+\xi_1)^{2\gamma}(1+\xi_1+\xi_0\xi_1')^\gamma}\right]$$
$$+(3\gamma-1)\xi_0\omega^2\xi_1\sin^2\theta+\gamma\xi_0^2\omega^2\xi_1'\sin^2\theta$$
$$-\frac{3\mu^2-1}{2GM(\xi_0)}g_0a_0(a^2+aa_0+a_0^2)\omega^2\xi_1. \quad . \quad . \quad . \quad (48')$$

Proceeding as in Case 2 above, we obtain the following differential equation in the place of (56) :

$$a_1''+\left[4-\nu+\omega^2(1-\mu^2)\frac{\xi_0^2\rho_0}{P_0}\right]\frac{a_1'}{\xi_0}+\frac{\rho_0}{P_0}\left[\frac{n^2}{\gamma}-\frac{\alpha g_0}{\xi_0}+\omega^2(1-\mu^2)\left(3-\frac{1}{\gamma}\right)\right.$$
$$\left.-\frac{3\mu^2-1}{2GM(\xi_0)}\cdot\frac{g_0}{\xi_0\gamma}a_0(a^2+aa_0+a_0^2)\omega^2\right]a_1=0, \quad . \quad . \quad . \quad (56')$$

where $\mu=\cos\theta$ and ν and α are as in (59).

Equations (57) and (58) remain unchanged.

It is apparent from (56′) that the period of oscillation will in general be a function of θ, and hence that the oscillations will not be only radial. In view of the importance of homogeneity established in our researches,[12] we will work out in detail the case of the homogeneous spheroid.

With the usual notation we have for the homogeneous spheroid:

V_0 = potential at an internal point (ξ_0, θ, ϕ)

$$=\tfrac{2}{3}\pi G\rho_0[3\,(1-\tfrac{1}{3}e^2)\,a^2-(1-\tfrac{1}{5}e^2)\,\xi_0{}^2-\tfrac{3}{5}\,\xi_0{}^2\,e^2\,\mu^2], \qquad (61)$$

$$g_0=\frac{4\pi G}{3}\,\rho_0\,c_1\,\xi_0, \qquad (62)$$

and $$P_0=\frac{2\pi G}{3}\,\rho_0{}^2\,c_2\,(r^2-\xi_0{}^2), \qquad (63)$$

where

$$c_1=1-\tfrac{1}{5}e^2+\tfrac{3}{5}e^2\,\mu^2, \qquad (64)$$

$$c_2=1-\tfrac{1}{5}e^2+\tfrac{3}{5}e^2\,\mu^2-\frac{3\omega^2\,(1-\mu^2)}{4\pi G\rho_0}, \qquad (65)$$

a = semi-major axis of the spheroid,

r = radius vector to the surface in the direction (θ, ϕ),

and $\mu=\cos\theta$.

The angular velocity ω is connected with the meridional eccentricity e by the relation[8]:

$$\frac{\omega^2}{2\pi\,G\rho_0}=\frac{3-2e^2}{e^3}\,(1-e^2)^{\frac{1}{2}}\,\sin^{-1}e-3\left(\frac{1}{e^2}-1\right)$$

$$=\frac{4}{15}e^2, \qquad (66)$$

neglecting the square of the ellipticity as defined in (6).

Putting $\xi_0=rx$ and substituting for g_0 and P_0 from (62) and (63), the differential equation (56′) reduces to

$$(1-x^2)\,\frac{d^2a_1}{dx^2}+\left(\frac{4}{x}-Ax\right)\frac{da_1}{dx}+Ba_1=0, \qquad (67)$$

where

$$A=4+2\frac{c_1}{c_2}-\frac{3\omega^2\,(1-\mu^2)}{2\pi G\rho_0}, \qquad (68)$$

and

$$B=\frac{3n^2}{2\pi\,G\gamma\rho_0\,c_2}-\frac{2\alpha\,c_1}{c_2}+\frac{3\omega^2\,(1-\mu^2)\left(3-\frac{1}{\gamma}\right)}{2\pi\,G\rho_0}$$

$$-\frac{9}{4\pi\,G\rho_0}\cdot\frac{\omega^2}{\gamma}\,(3\mu^2-1). \qquad (69)$$

The differential equation (67) has a regular singularity[14] at the origin. We find the roots of the indicial equation[14] to be 0 and -3. Taking the first root (as the second leads to an infinite value at the origin), we assume the following series solution for (67) :

$$a_1 = \sum_0^\infty b_\lambda \, x^\lambda \qquad \ldots \ldots \qquad (70)$$

On substitution we find that the coefficients of the odd powers of x vanish and those of the even powers satisfy the recurrence formula :

$$b_{2\lambda+2} = \frac{2\lambda(2\lambda-1)+2\lambda A - B}{(2\lambda+2)(2\lambda+5)} b_{2\lambda}, \qquad \ldots \ldots \qquad (71)$$

where λ is zero or a positive integer.

The series (70) will be a finite polynomial if

$$B = 2\lambda A + 2\lambda(2\lambda-1) \qquad \ldots \ldots \qquad (72)$$

If (72) be not satisfied, the series (70) will not terminate and can be shown to be convergent for $x<1$ but divergent for $x=1$ (that is, on the surface of the spheroid). Hence, by the extension of Abel's theorem[8] to series divergent on the circle of convergence, we shall have the limit of the amplitude infinite on the surface.

Therefore, all the modes of oscillation will be given by (72).

From (68), (69) and (72) we have

$$\frac{3n^2}{2\pi G\rho_0\gamma} = 2(2\lambda+\alpha)c_1 + 2(2\lambda^2+3\lambda)c_2 - \frac{3\omega^2}{2\pi G\rho_0}(1-\mu^2)\left((2\lambda - \frac{1}{\gamma} + 3 \right)$$

$$+ \frac{9}{4\gamma} \cdot \frac{\omega^2}{\pi G\rho_0}(3\mu^2-1) \qquad \ldots \ldots \qquad (73)$$

In order that the period may be independent of μ, we must have the coefficient of μ^2 in (73) equal to zero. From this, with the help of (64), (65) and (66) we obtain

$$\gamma = \frac{1}{2\lambda^2+5\lambda+3} \qquad \ldots \ldots \qquad (74)$$

As λ has to be integral or zero, the maximum value of γ given by (74) is 1/3. This is very much less than the critical value 4/3 for stability.[7] Further, from (73) and (74) we have

$$\frac{n^2}{\pi G\rho_0} = -4. \qquad \ldots \ldots \qquad (75)$$

The period of oscillation obtained from (75) is imaginary. We conclude, therefore, that a homogeneous rotating oblate spheroid of small ellipticity cannot execute purely radial oscillations of large amplitude.

As non-radial oscillations will meet with considerable material viscosity they will presumably not last long. This sets a mechanical limit to the amplitude of the pulsations. Eddington has given a different explanation.[4]

The author considers it a great privilege to record his grateful thanks to Professor A. C. Banerji, under whose guidance he has carried out the above investigation.

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ABSTRACTS

SECTION A

Radial Oscillations of a Variable Star. *By H. K. Sen*, Mathematics Department, Allahabad University.

In a recent and important paper, Prof. A. C. Banerji has shown that no radial modes of oscillation of large amplitude are possible for a Variable Star. Adopting Prof. Banerji's method, it has been shown in this paper that no radial mode of oscillation (even of small amplitude) is possible for a sphere with a homogeneous, central core, and density at any point in the annulus varying inversely as the p^{th} power of the distance of the point from the centre. Instability of radial oscillations has also been shown for the following models: (i) a sphere with vanishingly small core, and density varying inversely as the p^{th} power of the distance from the centre ($p > 3$, and amplitude taken to be small); and (ii) a sphere in which almost the whole mass is concentrated at the centre, oscillating with large amplitude. The conclusion has been drawn that, if the density vary inversely as the p^{th} power of the distance from the centre, no radial mode of oscillation is possible except for Dr. Sterne's models (i) the homogeneous sphere and (ii) the sphere in which the density varies inversely as the square of the distance from the centre.

Polytropic Gas Spheres with Variable Index. *By H. K. Sen*, Mathematics Department, Allahabad University.

Eddington's problem, of how far the properties of the variable polytrope lie between those of the limiting, uniform polytropes of maximum and minimum polytropic indices, has been considered. It has been shown from quite general considerations that the gravitational potential energy cannot be on isolated extremal property exhibited by the one-phase model of the limiting polytrope. Using Candler's equations, several intermediate properties have been deduced for the variable polytrope, besides those derived by Candler, in particular, the ratio of the central to the mean density. The temperature distribution of the variable polytrope has been considered in diverse aspects, and several integral theorems, as well as the monotonic decrease from centre to surface of certain physical variables, which is assumed for real stars, has been shown to follow from the polytropic equation.

Solutions of the Differential Equations, $f^r(x) = f\left(\pm \frac{1}{x}\right)$, where $f\left(\pm \frac{1}{x}\right)$ are properly defined. *By Santi Ram Mukherji*, Mathematics Department, Allahabad University.

The paper is just a generalisation of Silberstein's paper published in the *Philosophieal Magazine*, September, 1940. Cases beginning from $r=1$ to $r=6$ have been taken and some properties of the Differential Equations in the general case have been discussed.

SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS ARISING IN PROBLEMS OF VARYING VISCOSITY IN HYDRODYNAMICS. *By Santi Ram Mukherji*, Mathematics Department, Allahabad University.

The paper deals with the solutions of the following six differential equations which are required for finding the motion of incompressible fluid with varying co-efficient of Viscosity.

$$(i)\ \frac{\partial \zeta}{\partial t} = (\nu_0+\beta_1 x)\nabla^2\zeta, \qquad (ii)\ kr\nabla^2\zeta = \frac{\partial \zeta}{\partial t},$$

$$(iii)\ kr^2\nabla^2\zeta = \frac{\partial \zeta}{\partial t}, \qquad (iv)\ (c+kr)\nabla^2\zeta = \frac{\partial \zeta}{\partial t},$$

$$(v)\ (a_0+b_0 x)\nabla^2\zeta + c_0\frac{\partial \zeta}{\partial x} = 0, \qquad (vi)\ \frac{\partial V}{\partial t} = (a_0+a_1 x+a_2 y+a_3 z)\nabla^2 V.$$

Where $\nu_0, \beta_1, a_0, b_0, c_0, a_1, a_2, a_3, k$ are definite constants.

MOTION OF AN INCOMPRESSIBLE FLUID WITH VARYING CO-EFFICIENT OF VISCOSITY GIVEN BY $\mu=\mu_0+\epsilon_1 x$, WHERE ϵ_1 IS SMALL. *By Santi Ram Mukherji*, Mathematics Department, Allahabad University.

In this paper ϵ_1 has been taken so small that its square and higher powers are neglected and terms of the first order of quantities only have been retained. Motion of the fluid at a finite distance and at a great distance from the origin has been considered.

ON THE THEORY OF SPIRAL NEBULA. *By Brij Basi Lal*, Mathematics Department, Allahabad University.

In this paper (a) the possibility of the formation of nebulæ of irregular shapes has been considered; (b) the motion of the ejected particles along the arms of a spiral nebula has been investigated; and (c) the necessary condition for the formation of the spiral arms in a resisting medium has been found.

ON THE STRUCTURE OF THE K RADIATION OF OXYGEN. *By Dr. D. B. Deodhar and Dr. U. K. Bose*, Physics Department, Lucknow University.

In the investigations on soft X-rays of various metals with a two metre concave grating vacuum spectrograph the authors had occasionally to use the metals in the form of oxides which were rubbed upon the anticathode of the X-ray tube.

The spectograms showed lines due to the metals together with the oxygen lines. With a view to look for the origin of these oxygen lines the authors took microphotograph records of the Oxygen K line obtained on their plates. These microphotographs possessed a structure exactly similar to one observed by O'Bryan and Skinner (Proc. Ry. Soc. A Vol. 176, 1940) for oxygen in Oxides. The oxygen lines examined by the authors on their plates must have been therefore emitted by Oxygen atoms in the state of combination as oxides and not by free oxygen atoms.

Improvement of Nitrogen Status of Soils. *By N. R. Dhar*, Indian Institute of Soil Science, Allahabad.

In a communication to the Nature (Nature 138, 1060, 1936) it was reported that farm-yard manure (cowdung) when added to soil fixes the atmospheric nitrogen and that the value of cowdung lies not only in its nitrogen content but also in its power to fix atmospheric nitrogen. Field trials have confirmed this observation. That nitrogen fixation or accumulation on the addition of farm-yard manure takes place even in soils of temperate climate is evident from the following results obtained from the classical field trials at Rothamsted:—

	Results obtained in 1926 Total nitrogen
(1) Receiving no manure since 1845	0·095%
(2) Receiving farm-yard manure since 1852	0·095%
(3) Receiving complete artificials and $(NH_4)_2SO_4$ ...	0·099%
(4) Receiving complete artificials and farm-yard manure ..	0·253%
(5) Receiving potash and phosphate but no nitrogen ...	0·090%

Our results show that the loss of nitrogen when ammunium sulphate is added to the soil is minimised by the addition of carbonaceous substances like molasses, hay, cowdung, leaves etc., which act as negative catalysts in the process of nitrification leading to a loss of nitrogen mainly in the gaseous state.

Russell ("Soil Conditions and Plant Growth, 1931, page 362) has reported that the nitrogen content of a grass-land increases from 0·152% in 1856 to 0·338% in 1912. Similarly, a land permanently covered with vegetation for 24 years showed an increase of total nitrogen from 0·108 to 0·145%.

The foregoing observations clearly show that carbonaceous substances help in the accumulation of nitrogen and its fixation and this explains why organic manures are valuable in steadying crop-yields. Not only the total nitrogen but the available nitrogen is also increased in the soils by this process.

The residual effect of stable manure as observed throughout the world may be due not mainly to the conservation of nitrogen as hitherto believed but is caused by the fixation of atmospheric nitrogen through the oxidation of energy materials like pentosans, celluloses, fats, etc. It seems that wherever a residual effect of a manure has been observed, *e.g.*, with hay or stable-manure, or molasses, it is, perhaps, chiefly due to nitrogen fixation in the soil and no residual effect will be observed with a manure which is incapable of fixing atmospheric nitrogen although it may contain carbon. This viewpoint is supported by the observation of Morse (Mass Agric. Exp. Sta. Bull., 1936, No. 333) which shows that with a nitrogen status of soils approximately 0·15% there is no nitrogen accumulation with legumes. The green manures when added to the soil slowly undergo oxidation with loss of carbonaceous substances and nitrogenous compounds. The residual alkali intensifies the loss of nitrogen from such soils and hence the nitrogen status of soils is not improved as with farm-yard manures or molasses.

It has already been reported by us that in general the available nitrogen in tropical soils is much greater (100——200 lbs. per acre) than in soils of temperate countries (20——40 lbs per acre) and that is why a crop can be drawn in tropical soils in about 4 months whilst in non-tropical countries about 8-9 months are needed, although the total nitrogen in tropical soils is less than that in soils of temperate countries.

In temperate climates an attempt should be made to improve the available nitrogen status by ploughing the soil and exposing it to light and air in the spring, summer or autumn when the sunlight is strong. Instead of adding ammonium sulphate to soils in temperate countries to increase the available nitrogen it may perhaps be less expensive to increase ploughing and breaking

up of the soils, making conditions more favourable for oxidation and obtaining a better crop-yield without making the soil more acidic, as happens on the addition of ammonium sulphate. In the case of soils in temperate climates which have deteriorated and may have been given up for the purpose of cultivation it seems that the remedy lies in the addition of more cow-manure (farm-yard manure) or hay or other readily decomposable carbonaceous substances like molasses but not by the addition of legumes, which have no residual effect on such soils.

LOSS OF NITROGEN AND ITS RETARDATION UNDER STERILE CONDITIONS. *By N. R. Dhar and N. N. Pant.* Soil Science Institute, Allahabad.

The loss of nitrogen from urea and gelatine and in absence or presence of sugar has been investigated under completely sterile conditions when mixed with sterile soils or oxides like ZnO, TiO_2 etc. both in light and in the dark. It has been observed that even under completely sterile conditions and without any bacterial infection during the course of the experiment, there is a greater loss of nitrogen in light than in the dark and the presence of sugar markedly retards this loss. When the amount of the nitrogenous compound is not high there is appreciable fixation of nitrogen in presence of sugar. The explanation of Doryland based on the energy requirements of micro organisms (N. D. Agri. Exp. Sta. Bull., 116, 1916) is untenable in these cases and the experimental observations are easily understood from the viewpoint that just as in the case of metabolism in the animal body carbohydrates act as negative catalysts in the oxidation of proteins and can preserve the body or exogenous proteins, similarly, in the soil carbohydrates conserve protein or other nitrogenous compounds by acting as negative catalysts in the oxidation of such nitrogenous compounds. It appears that both in the soil and in the animal body oxidation processes are controlled by physico-chemical laws, as in ordinary oxidation reactions involving negative catalysis.

INVESTIGATIONS ON ALUMINIUM SILICATE SOLS. *By S. Ghosh and S. P. Srivastava,* Chemistry Department, Allahabad University.

(1) Sufficiently concentrated Sols of Aluminium Silicate containing varying amounts of Aluminium Oxide and Silica were prepared.

(2) The compositions of the different sols were between the limits of $\frac{SiO_2}{Al_2O_3} = 1{\cdot}6$ and $\frac{SiO_2}{Al_2O_3} = 0{\cdot}8$. All attempts to prepare sols containing greater or smaller SiO_2 and Al_2O_3 ratio than these values failed.

(3) All the sols containing different proportions of SiO_2 and Al_2O_3 were positively charged. Charge reversal was possible by negatively charged humic acid Sol.

(4) The sols containing greater proportions of silica yielded better gels when coagulated by monovalent anions than the sols containing less of silica and coagulated by bivalent anions.

(5) All the sols on coagulation showed (*a*) abnormal behaviour on dilution towards KCl, (*b*) ionic antagonism by mixture of KCl and other polyvalent anions and (*c*) the phenomenon of acclimatization with KCl.

With $CaCl_2$ abnormal dilution effect was developed only with the sol containing largest proportion of silica. The coagulating power of $CaCl_2$ was also least with this sol.

(6) The above results on coagulation can be explained by the views of Ghosh and Dhar from the adsorption of similarly charged ions by the colloid particles.

(7) Adsorption experiments with the different sols show that they are capable of adsorbing both the anions and cations from the added electrolytes. In general the sols containing greater amounts of silica adsorb more of cations than the sols containing less of silica, whilst the adsorption of anions increases with the increasing Al_2O_3 content of the sols.

(8) The base or acid exchange has been found to be sufficiently a slow process.

(9) These sols have been shown to be different from the sols obtained by mixing aluminium hydroxide and silicic acid sols prepared separately.

(10) The adsorption capacity of clay for both cations and anions is due to its inorganic constituent, *viz.*, silica and basic oxides. The humus considerably modifies the physical character of clay and increases its capacity to adsorb the cations.

CHEMICAL EXAMINATION OF THE SEEDS OF *Nigella Sativa* Linn. (Magrel) part I. Fatty Oil By Bawa Kartar Singh and Ram Das Tewari, Chemistry Department, Allahabad University.

In this paper the fatty oil from the seeds of *Nigella Sativa* Linn. (*Magrel*) has been examined and found to contain the glycerides of oleic, linolic, myristic, palmitic and stearic acids, the percentages of which are given below:—

Oleic	35·99
Linolic	44·45
Myristic	0·26
Palmitic	6·31
Stearic	2·45
Unsaponifiable (sterol)	0·03

It has been shown that the oil examined by Crossley and Le Sueur (Agri. Ledger India, 1899, No. XII, p. 34; 1911-12 p. 112) was not a volatile oil but a mixture of fatty and volatile oils. Further the diolefinic acid present in the oil of *Nigella sativa* is linolic acid in our case, whereas it is telfairic acid in the oil examined by Bures ant Mladkova, (Casopis ceskoslov Lekarnictva, 10, 317—323, 1930).

SECTION B

STUDIES ON THE PHOTOCHEMICAL ACTION IN PLANTS. (V) LIGHT RESPIRATION OF *EUGENIA JAMBOLANA* LEAVES AT DIFFERENT PERIODS OF STARVATION. *By Shri Ranjan and Suresh Chandra Tyagi*, Botany Department, Allahabad University.

Varying periods of previous starvation of leaves of *Eugenia jambolana*, show that the increase in light respiration is highest when the leaves are least starved. With prolong starvation the increase in light respiration decreases, till the tenth day of starvation. After the tenth day the respiration in light again increases. An analysis of the total carbohydrates shows a decrease till the tenth day and then increase. The amino-acids, on the other hand, increase up to the end of the sixth day and then decrease. Thus, some correlation has been established between the increase in light respiration and the carbohydrate content, but not with the amino-acids.

STUDIES ON THE PHOTOCHEMICAL ACTION IN PLANTS. (VI) THE EFFECT OF SOME COLOURED LIGHTS ON THE LIGHT RESPIRATION OF *EUGENIA JAMBOLANA* LEAVES. *By Shri Ranjan and Suresh Chandra Tyagi*, Botany Department, Allahabad University.

(1) The effect of monochromatic lights on the light respiration of green leaves has been investigated.

(2) Red light, which is essential for photosynthetic process, does not affect in any way the respiration rate of plants.

(3) Both the green and the blue lights increase the respiration rate of green leaves in light. The writers have shown that these lights do not affect in any way the photosynthetic process. They have thus divided the respiration of green plants into (a) dark respiration and (b) light respiration. Enzymatic activity alone is sufficient for the former, but for the latter blue, or green light is required. The rôle of the carotinoid pigments in photo-oxidative respiration is emphasised.

STUDIES ON THE LOSS OF FERTILITY BY CERTAIN FUNGI IN CULTURE. *By R. N. Tandon*, Botany Department, Allahabad University.

1. *Melanospora destruens*, *Phytophthora cactorum* and *Fusarium fructigenum* Strain A were maintained on a synthetic medium and on the same medium with the addition of growth producing substances (in the form of extract of lentils) and were subcultured at monthly intervals over a period of 15 months. Considerable variation was seen in each case with a tendency to a general deterioration in fertility but this could not be correlated with the medium on which the various culture lines were maintained. Work on *M. destruens* was continued for 3½ years and in general it was confirmed that the nature of medium on which the parent cultures were maintained had no correlation with these deteriorations.

2. *M. destruens* was also maintained on various modifications of the synthetic medium. The resulting cultures showed considerable variations which were largely independent of the medium on which the fungus was maintained.

3. No difference could be seen between the amount of variation in culture lines subcultured at intervals of one or of three months.

4. The type of inoculum (whether spores alone or mycelium alone) did not influence the fertility of the cultures of *M. destruens*.

5. Variation was seen in cultures developing from parts of the same hypha or from spores from the same perithecium of *M. destruens*.

6. Abundant sector formation has been shown by Malanospora cultures, pointing to the existence of strains of different intrinsic sporulating capacity.

7. Continuous subculturing of *M. destruens* over a period of 3½ years shows that in mass subculturing many poorly sporing or sterile parent cultures give rise to strongly sporing and very fertile cultures. These may remain fertile or may begin to deteriorate.

SEGMENTATION OF CAUDAL SUCKER OF THE ARHYNCHOBDELLID LEECHES. *By M. L. Bhatia*, Zoology Department, Lucknow University.

There has been a good deal of controversy about the exact number of segments which form the caudal sucker of Arhynchobdellid leeches. A detailed study of a few genera belonging to this group, supported by the conditions seen in the embryos of *Hirudinaria*, reveal that *seven* segments form the caudal sucker of the Arhynchobdellid leeches.

CYTOPLASMIC INCLUSIONS IN THE OOGENESIS OF *TURDOIDES TERRICOLOR TERRICOLOR*. *By D. N. Varma*, Zoology Department, Allahabad University.

The work on the cytoplasmic inclusions in the Oogenesis of birds, from the point of view of modern cytological technique, is meagre. The only papers, published after 1914, are those of Brambell in 1925, Das in 1931, Srivastava in 1933 and the latest is that of Singh in 1938. In Seven Sisters, as this bird is popularly called, the cycles of Golgi and mitochondrial bodies were traced. The phenomenon of infiltration of both Golgi bodies and mitochondria from the follicle cells to the oocyte took place in a haphazard manner and continued till the formation of Zona radiata while yolk formation took place at quite an early stage——it was even seen in the Yolk nucleus of Balbiani stage. Golgi bodies were seen to be responsible for the formation of fatty yolk and the elaboration of the albuminous yolk was found to be of seasonal occurrence. Polynuclearity was a phenomenon most extensively observed in this bird.

LIBERATION OF SEXUAL ELEMENTS IN *MARPHYSA MOSSAMBICA* PETERS. *By Najm-ud-Din Aziz*, Lahore.

In transverse sections of a mature female Polychaete belonging to the species *Marphysa mossambica* Peters, a group of well-developed eggs was seen in the cœlomic cavity. The author describes the course followed by the eggs, and discusses the mode of liberation of the sexual elements.

A COLLECTION OF OLIGOCHÆTES FROM SOME HIGH MOUNTAIN LAKES IN KASHMIR. *By Najm-ud-Din Aziz*, Lahore.

The present paper contains a preliminary account of some Oligochætes collected from 17 lakes at about 12000 ft., *viz.*, Sheshnag, Handil Sar, Sona Sar, (1); Sona Sar (2), Duodhnag, Tar Sar, Chand Sar, Tulian, Harnag, Khem Sar, Yam Sar, Gangabal, Nandkol, Kul, Vishan Sar, Kishan Sar and Gad Sar. The collection includes members of the aquatic family Naididæ and of the terrestrial families Moniligastridæ and Megascolicidæ. The author also discusses the geographical distribution of these Oligochætes.

ENTOMOSTRACA FROM SOME HIGH MOUNTAIN LAKES IN KASHMIR. *By Guran Lal Arora*, Lahore.

In the present paper the author gives a short account of the Entomostraca collected from ten lakes, *viz.*, Tulian, Harnag, Yam Sar, Khem Sar, Vishan Sar, Kishan Sar, Gad Sar, Gangabal, Kul and Nandkol, all situated at about 12000 ft. and fed by water from the glaciers above.

The Entomostraca include Cladocera belonging to the families Daphnidæ, Chydoridæ and Macrothridæ; Ostracoda belonging to the family Cyprididæ; and Copepoda belonging to the families Centropagidæ and Cyclopidæ. It is interesting to note that in the clear water of these lakes, at such great altitudes the Copepoda were deep red or orange, the Ostracods green and the Cladocerans dark brown, light brown or white in colour.

A PRELIMINARY REPORT ON SOME AQUATIC INSECTS FROM KASHMIR. *By D. R. Puri*, Lahore.

The collection includes approximately forty species belonging to the orders Odonata, Ephemeroptera, Coleoptera, Trichoptera and Diptera.

The order Odonata is represented by about twelve species, which were collected mostly from the Liddar Valley at altitudes ranging from 7000 ft. to 9000 ft A few species were restricted to particular areas, while others were more widely distributed.

Three species of May-flies (Ephemeroptera) were collected from Kishan Sar and Nand Kol lakes.

The order Coleoptera includes an amphibious beetle from Gangabal lake. It floats on the surface of water, but frequently flies to the land.

There are about fifteen species of caddis flies (Trichoptera) in the collection. All species, except one, were taken on light.

Three species of crane-flies (Diptera) were collected from below Tulian lake (11000 ft.). One of them is exceptionally large, each wing having an expanse of 3 c. m.

SOME SPIDERS FROM KASHMIR. *By Sukh Dyal*, Lahore.

The present paper deals with *Campostichmma* sp. of the family Agelenidæ collected from Vishan Sar (12000 ft.) and Kishan Sar (12500 ft.) lakes. These spiders, black in colour, were found near the banks of the lakes. Another spider, *Ocyale* sp. of the family Lycosidæ, dark brown in colour, was found on the banks of Sheshnag lake (11700 ft.). Two specimens of *Lycosa* sp. and one of *Araneus* were collected from Pahalgam (7000 ft.). A specimen of *Nepohila* was found at Chandanwari (8000 ft.).

A SPECIES OF INDIGENOUS FISH FROM GAD SAR LAKE, KASHMIR. *By Nazir Ahmad*, Lahore.

This paper is a preliminary report on the ecology and systematic position of an indigenous fish obtained in the summer of 1941. The fish were small in size and mottled like *Salmo fario* Linnæus. These were very active and were seen to move in shoals.

In the paper an account of the adult and fry is given.

ON A SMALL COLLECTION OF VERTEBRATES FROM HIGH ALTITUDES. *By Nazir Ahmad*, Lahore.

The collection under report was made during the summer of 1940 and 1941, from Kashmir, and comprises tadpole and adult of the toad, *Cophophryne sikkimensis* (Blyth); young and adult of the

lizard, *Leiolopisma himalayanum* (Günther); a pit viper, *Ancistrodon himalayanus* (Günther), and two rodents, *Alactaga* sp. and *Lagomys roylei* Ogilby. The present paper contains an account of the systematics and ecology of the above forms.

THE "YOLK NUCLEUS OF BALBIANI" IN THE SPIDER *LYCOSA PUNETIPES*. *By Ram Saran Das*, Zoology Department, Allahabad University.

1. The yolk nucleus arises in the early oocytes in a juxtanuclear position as a hollow vesicle traversed by numerous fine intercrossing fibres.

2. In older oocytes this vesicle is surrounded on all sides by a lamellated membrane of varying thickness. This does not consist of mitochondria, but on the contrary is purely a product of cytoplasmic differentiation.

3. The Golgi bodies are fine granular elements which completely encircle the yolk nucleus. Between the outer margin of the yolk nucleus and the Golgi layer there occurs a clear zone almost completely devoid of the inclusions.

4. Vitellogenesis does not appear to be related to the yolk nucleus. The yolk nucleus persists in the vicinity of the principal nucleus in fairly advanced eggs, whereas vitellogenesis commences on the periphery.

THE GOLGI BODIES AND THE SECRETION OF FAT DROPLETS IN THE EGGS OF CERTAIN ANIMALS. *By Murli Dhar Lal Srivastava*, Zoology Department, Allahabad University.

The paper embodies the results of an attempt to investigate the relationship of the Golgi elements and fat droplets in the eggs of certain animals by means of an osmium-silver technique. The ovary is fixed in 1·5% osmic acid for twenty-four hours and subsequently impregnated with silver and treated with hydroquinone as in Cajal's method. Sections are mounted in Canada balsam directly or after extracting the fat in turpentine.

ON THE STRUCTURE AND ORIGIN OF THE CORPUS LUTEUM IN THE LIZARD *HEMIDACTYLUS FLAVIVIRIDIS* (RUPPEL). *By S. K. Dutta*, Zoology Department, Allahabad University.

There has been a fairly good amount of work on the female reproductive cycle in the class Mammalia, in which most of the study is directed to the behaviour of the corpus luteum of the ovary. But the changes undergone by the ruptured follicle have not been so intensively studied in the lower Vertebrata. This paper deals with the structural variations of the ovary in the lizard *Hamidactylus flaviviridis* (*Ruppel*). The occurrence of a corpus luteum is recorded and its histological structure described.

STUDIES ON THE SIX NEW SPECIES OF THE GENUS *NEODIPLOSTOMUM*, RAILLIET, 1919 (family *DIPLOSTOMIDÆ* POIRIER, 1886). *By P. N. Chatterji*, Zoology Department, Allahabad University.

This paper gives a description of the six new species of the genus *Neodiplostomum*, Railliet 1919 (Family *Diplostomidæ*, Poirier 1886). These species were collected from the common kite, *Buteo rufinus rufinus*, hawk, *Accipetres nisus malanoschistus*, woodpecker, *Brachypternus bengalensis bengalensis* and

Indian Koel, *Eudynamis scolopaceous* caught from the different villages near Allahabad. According to the subdivision of the genus into two subgenera by Dubois 1937 three species are included in the subgenus *Neodiplostomum*, Dubois, 1937 and the three in the subgenus *Conodiplostomum*, Dubois, 1937. The diagnostic characters of the species are given and their relationships discussed. *Neodiplostomum eudynamis* n. sp. differs remarkably from all the other species of the genus on account of the presence of three small muscular papillæ on the ventral side of the body just in front of the genital atrium. In *Neodiplostomum nisus* n. sp. the genital cone is absent, but a genital papillæ, which lies just in front of the opening of the hermaphroditic canal in the genital atrium is present.

Some Fresh-water Fishes and Fisheries of the United Provinces. *By Dr. A. J. Faruqi*, Zoology Department, Agra College, Agra.

The paper is divided into three parts. The first part deals with the four different varieties of nets and their different modifications which are used in the various parts of the province to suit varying conditions of rivers, lakes and other reservoirs of water. The following four types of nets are described :—

1. Trap-net,
2. Hand-net,
3. Drag-net, and
4. Fixed-net.

The description of each type of net includes full particulars regarding its construction and the method of using it.

The second part of the paper deals with 69 species of fishes. In each case interesting features such as market value, the time of the year when it is most abundantly caught and migratory instinct if possessed are dealt with in detail. A list of fishes with vernacular names is given at the end of the second part.

The third part deals with a general note regarding the scope of developing inland-fisheries in the Province. It is discouraging to note that the Local Government of the Province has so far adopted an indifferent attitude towards this problem whereas other Provinces have been more active in the field and fruitful results are seen in them. One of the major recommendations deals with the suggestion regarding the conversion of the Kitham Reservoir (Agra) into a piscine culture station and the establishment of another big station at Narora falls (Bulandshahr). On the eastern side of the Province, natural reservoirs are in abundance and these may be utilised for stocking fishes.

An elaborate scheme for the establishment of a Fishery Department is possible only if the Local Government become interested in the question and it is hoped that once the Department is started, it will become self-supporting after a few years.

Fresh-Water Polyzoa from High Mountain Lakes in Kashmir. *By N. K. Gupta*, Lahore.

In the present paper a preliminary account of some of the encrusting species of Polyzoa probably belonging to the genus *Plumatella* Lamark, from two lakes of Kashmir, namely, Ganga Bal (11714 ft.) and Nand Kol (11200 ft.) is given. At the time of collection the following aquatic temperatures were recorded : Ganga Bal 64°F, Nand Kol 59°F. It is interesting to note that no trace of any Polyzoa was found in any of the other fifteen high mountain lakes, whereas it was very common in the above-mentioned lakes.

Address by Dr. PANNA LALL, M.A., B.Sc., LL.B., D.Litt., C.I.E., I.C.S., Adviser to His Excellency the Governor, United Provinces, at the Eleventh Annual Session of the National Academy of Sciences, India, held at Agra on February 13, 1942.

MR. PRESIDENT AND MEMBERS OF THE NATIONAL ACADEMY OF SCIENCES :

HIS EXCELLENCY the Governor of these provinces has entrusted me with a message, which I should first deliver to you.

His Excellency's Message—

"Two years ago I had the honour to attend the annual meeting of the National Academy of Sciences and I sincerely regret that it has not been possible for me to attend the meeting at Agra this year. Two years ago I spoke of the mad policy of one nation, or perhaps more accurately, of one man which, apart from its more direct results, has had the indirect result of handicapping throughout the world scientific investigation, and which has turned the ingenuity of man to inventing means of destroying life rather than of preserving and improving it. Since then the conflict has spread throughout the world, but a time

> will come again when swords can again be beaten into ploughshares and spears into pruning hooks and it is well that we should maintain the continuity of scientific thought designed for other purposes than human destruction. The National Academy of Sciences is designed to secure that continuity and I wish its meeting this year every success. I am glad to say that the Provincial Government has found itself able to give a recurring grant this year to the Academy in place of the non-recurring grants given in previous years and I know that this contribution to your resources will be well repaid."

I thank you for the honour you have conferred upon me by inviting me to inaugurate this session. The importance and weight which attaches to this Academy is evident from the list of distinguished scientists who are its members and from the galaxy of eminent men which I see resplendent here today. It is a pleasant coïncidence that you are meeting today in this city of Agra where I spent many happy years of my academic life.

It would indeed be presumption for one, with my slender equipment, to speak in such an assemblage either on the value of the work which you have done or to say to you what you should do; or again to attempt to dogmatize on the charge which is sometimes levelled against Science that

it is responsible for the misery and injury which has been inflicted by the warring nations and that therefore its mischievous activities should be curbed, if it should not be altogether anaesthetized and laid at rest for a time, like a lunatic in a padded cell. But I cannot refrain from taking advantage of this opportunity to make a few observations. My two immediate predecessors, in the role which I fill today, referred (with the humility of the real savant) to the exclusively classical education which they had received. Unlike them I have had the unusual good fortune of receiving a classical as well as scientific education. And though that justifiably labels me as jack of both, yet it gives me an advantage also. For without being open to the charge of partiality to science, I can appreciate, and offer my tribute of admiration to, the valuable work which has been done by your members in expanding the horizons of knowledge; dealing on the one hand with the immensities of the suns and the stars; and on the other with the unimaginable minuteness of nuclei and electrons. But I would not be so foolish as to go into any further details.

Science and the horrors of war—

The statement that Science is to be held responsible for the suffering and misery which has been and is being inflicted this minute on our fellow-men—but from which

Providence has so far saved us—is one which I never thought would need a solemn and serious refutation, until latterly when I heard that view expressed by responsible speakers whom I hold in great regard and esteem. I cannot brush this aside as cheap or prejudiced criticism. I would, therefore, take this opportunity, with your permission, to submit a few considerations to the contrary, although they must needs appear to you obvious and even elementary. While not yielding to any in my love for classics and the fine arts, I am utterly unable to join hands with those who would condemn science as the source of our misfortunes and would crush its activities. This unjust attitude has recently provoked a scientist to retort that the Government of the world should be handed over to scientists who might achieve greater success than the politicians in stopping this awful succession of wars. Without subscribing to that extreme demand, I would say, with the full knowledge of the horrors of the past two and a half years, that we still want science and more science—pure science; applied science; technical science. The fact that the knowledge acquired by science has been put by some to unworthy uses and has resulted in human suffering may denote a failure of religion or philosophy, but is surely not sufficient to prove the iniquity of scientific pursuit. It should

rather furnish a reason for educating those unworthy persons in the proper ideals of human conduct—in other words—Moral Science. Human passions were not less brutal before the advent of electricity and radio-activity. Humanity exhausted all its resources in devising the lingering tortures of the poisoned arrow and daggers which opened inside the body with expanding jagged edges. And right back in the dawn of our race, how could our first ancestors have succeeded in establishing their supremacy over the elephant or the ~~long~~-toothed tiger without the use of weapons fashioned by the scientific knowledge then available? Would you condemn the discoverer of iron or copper for the use of the battle axe; or blame the stone club which the early male employed with such success to strengthen his matrimonial suit? But for patient researches in the properties of stone, iron and copper, there would have been little of our boasted civilization. And quite apart from the visible and tangible enemies with which our forbears had to contend, science has given us information about a host of others, too minute to be seen or felt, which nevertheless are deadly. The typhoid or the cholera germ is no less deadly than the karait or the boa-constrictor. It is science alone which can discover for us, first the existence of these enemies and next the way of checkmating them. We must have, can have, nothing

but praise and encouragement for scientific research. The tragedy rather is that knowledge is co-existent with power, and Providence has given us free will to employ that power as we list. That power or energy—Shakti—has a dual aspect, one gentle and the other fierce—the *Gauri* and the *Kali* —and Man, the supreme purpose of evolution, has yet to learn how to understand and propitiate both.

In our ancient books there is a story of a research student eager to acquire mastery over the forces of nature, who obtained the power to burn up anything which he touched. And immediately he tried to employ that knowledge on the very person to whom he owed it. But Providence could not tolerate such wickedness, and in the end all that he succeeded in achieving was to burn himself to ashes with his own power. May be that that immortal story is being enacted once again before our eyes.

Science and War Effort—

Our immediate concern is with the actual circumstances surrounding us. The War is there as wars have always been. Like a vast devastating prairie fire, it is continuing its relentless march and ours is the only great country that has so far escaped ; but its horrible tongues are even now licking our borders. Is this, I ask, the time to put science away in

cold storage, or is it the time when the clear duty of all scientists is to make the fullest use of their intellect and resources, and together to devise the most effective means of counteracting the menace that stares us in the face.

Gentlemen, bear with me when I pause and consider how helpless we should be without science. It is only with the help of science that the ships of the navy are still sailing on the High Seas, bringing us food, medicines, and machinery. Our valiant soldiers, whether in the Mid-East or the Far East, must maintain touch with us by ether, air, or sea. You will recall how the advent of the magnetic mine and the accoustic mine by the enemy paralysed our shipping and our communications for a brief while till scientists—you—came to our rescue. So too against each new diabolical engine of the enemy must you discover an effective defence, and before it becomes too late. The transport of troops ; the repatriation of the wounded and the sick ; the surgical treatment of the maimed and the mutilated would be impossible without the aid of science. It is to the scientist that the layman appeals for advice how to deal with the explosive, the incendiary, or the anti-personnel bomb. The long continuance of the war must inevitably cause a shortage of vital commodities. It is to you that the world will look for substitutes, alternatives, and synthetic products, for many

a natural substance. It is your researches which will teach us how to employ the available resources to the greatest good of the greatest number so that we may preserve our healths and stand up squarely to the attack. Without science, War effort will collapse like a castle of sand, and the gruesome spectre of Death and devastation will stalk this our fair land. With the help of science, we may be able to survive this catacylsm and after the war may be able to piece up such bits of civilization as may still be left and start building a new world with new hopes and new aspirations.

The Scientific attitude—

I have so far only referred to the objective facts of science, its discoveries of matter and energy and invention of machines. Great as has been the contribution of science to the happiness and welfare of mankind by these objective items, the debt which humanity owes to science is, however, still wider. The method of science has wrought a revolution in the entire realm of human thought and activity.

The scholar who inaugurated your session last year described the merits of classical education in these eloquent words—

> 'The severe mental discipline which it provided strengthened and invigorated the mind, taught us habits of accuracy and concentration, inspired us with an ardent love of

truth, enabled us to grasp the general principle implicit in a series of facts or phenomena, and instructed us in the processes of reasoning and logic. Indeed it did more ; for it did not neglect the emotional side of human life and it opened our eyes to the beauties of art and literature. '

A truly beautiful prospect ; but may I, with all respect, claim for Science almost all this and a great deal more—

The devaluation of authority and tradition ; the insistence on personal observation and experiment ; the search for unbiassed evidence ; tireless patience never losing hope ; the sorting out of data and conclusions and their orderly classification ; exactness of thought and expression ; the interest in causal relations as distinct from interest in things for their own sake ; the presentation of the results of experiment regardless of their consequence ; and a loyal and tenacious adherence to those results whether they fit it with the accepted view of things or not.

These are some of the constituents of the scientific attitude towards life. This attitude has silently but surely permeated our entire world. Its methods are with great advantage applied to such different realms as philosophy, economics, sociology, and even to red-tape administration. More, science has developed a new outlook upon our environment and a new reaction to it, making some things seem

more valuable and others not. We can no more ignore these new standards of science in a discussion of the general problem of values than we can ignore, say, the telephone or the Railway train. It has given us a new angle for viewing beauty and art, as modern paintings, poems and buildings, even the appointments of our homes bear out, with their dominant notes of precision, economy and exact finish. These are the gifts of Science.

It is by a synthesis to be effected by the scientist, the thinker and the artist, each bringing his own specialized contribution to the common pool, exploring the inmost recesses of human thought and activity, that our lives can be made fuller and richer and civilization truly advance.